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Applications of the KKM property to coincidence theorems, equilibrium problems, minimax inequalities and variational relation problems

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Abstract. In this paper, we establish coincidence-like results in the case when the values of the correspondences are not convex. In order to do this, we define a new type of correspondences, namely properly quasi-convex-like. Further, we apply the obtained theorems to solve equilibrium problems and to establish a minimax inequality. In the last part of the paper, we study the existence of solutions for generalized vector variational relation problems. Our analysis is based on the applications of the KKM principle. We establish existence theorems involving new hypothesis and we improve the results of some recent papers.

Key Words. KKM property, coincidence theorem; equilibrium problem; minimax inequality; variational relation problem.

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1 Introduction

The aim of this paper is twofold: firstly, to establish a Fan type geometric result and to apply it in order to obtain some coincidence-like theorems for the case when the images of the correspondences are not convex. Further, new theorems concerning the existence of solutions for equilibrium problems are provided. This study also aims to investigate whether the class of minimax inequalities can be extended. In fact, we obtain a new general minimax

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inequality of the following type: $\inf_{x\in X}\sup_{y\in Y}t(x,y) \leq \frac{\sup_{y\in Y}\inf_{z\in Z} q(y,z)}{\inf_{z\in Z}\sup_{x\in X} p(x,z)}$. Another recent result, due to the author, regarding minimax inequalities for dicontinuous correspondences, is [27].

In this first part of the article, the originality consists of introducing a new type of properly quasi-convex-like correspondences, which proved to play an important role in our results. The method of proof is based on the well known KKM property. There exists a large literature containing applications of the KKM property to coincidence theorems, equilibrium theorems, maximal element theorems and minimax inequalities. We refer the reader, for instance, to M. Balaj [2], Lin, Ansari and Wu [18], Lin and Wan [19] or Park [24].

Secondly, the paper explores how the KKM principle can promote new more theorems which show the existence of solutions for some classes of variational relation problems. We emphasize that, here, the method of application of the KKM property is new and provides new hypotheses for our resarch.

The rest of the article is organized as follows. In Section 2, we introduce notations and preliminary results. In Section 3, a convex-type property for correspondences is defined and some examples are given, as well. We use this type of correspondences to obtain coincidence-like theorems, to solve vector equilibrium problems and to establish a minimax inequality. In Section 4, we apply the KKM principle to general vector variational relation problems involving correspondences. Concluding remarks are presented in Section 5.

2 Preliminaries and notation

Throughout this paper, we shall use the following notations and definitions:

Let A be a subset of a vector space X, 2^A denotes the family of all subsets of A and co$A$ denotes the convex hull of A. If A is a subset of a topological space X, $\overline{A}$ denotes the closure of A in X.

If $T, G : X \to 2^Y$ are correspondences, then co$G$ and $G\cap T : X \to 2^Y$ are correspondences defined by $(\text{co}G)(x) :=\text{co}G(x)$ and $(G \cap T)(x) := G(x) \cap T(x)$, for each $x \in X$, respectively.

Given a correspondence $T : X \to 2^Y$, for each $x \in X$, the set $T(x)$ is called the *upper section* of T at x. For each $y \in Y$, the set $T^{-1}(y) := \{x \in X : y \in T(x)\}$ is called the *lower section* of T at y. The correspondence $T^{-1} : Y \to 2^X$, defined by $T^{-1}(y) = \{x \in X : y \in T(x)\}$ for $y \in Y$, is called the *(lower) inverse* of T.

For $A \subset X$, let $T(A) = \bigcup_{x\in A} T(x)$.

If X is a nonempty set and Y is a topological space, the correspondence $T : X \to 2^Y$ is said to be *transfer open-valued* [31] if for any $(x, y) \in X \times Y$ with $y \in T(x)$, there exists an $x' \in X$ such that $y \in\text{int}T(x')$. T is said to be *transfer closed-valued* [31] if for any $(x, y) \in X \times Y$ with $y \notin T(x)$, there

exists an $x' \in X$ such that $y \notin \overline{T(x')}$. The correspondence T is transfer closed-valued on X if and only if ([32]) $\cap_{x \in X} \overline{T}(x) = \cap_{x \in X} T(x)$.

Further, we present the following lemma (Propostion 1 in [17]).

Lemma 1 *Let Y be a nonempty set, X be a topological space and $T : X \to 2^Y$ be a correspondence. The following assertions are equivalent:*

a) $T^{-1} : Y \to 2X$ is transfer open-valued and T has nonempty values;
b) $X = \bigcup_{y \in Y} \mathrm{int} T^{-1}(y)$.

Notation We will denote by Δ_{n-1} the standard $(n-1)$-dimensional simplex in $\mathbb{R}^n$, that is, $\Delta_{n-1} := \left\{ (\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbb{R}^n : \sum\limits_{i=1}^{n} \lambda_i = 1 \text{ and } \lambda_i \geqslant 0, i = 1, 2, ..., n \right\}$.

Let $C^*(\Delta_{n-1}) := \{g = (g_1, g_2, ..., g_n) : \Delta_{n-1} \to \Delta_{n-1}$ where g_i is continuous, $g_i(1) = 1$ and $g_i(0) = 0$ for each $i \in \{1, 2, ..., n\}\}$.

In [26], we introduced the concept of a weakly naturally quasi-concave correspondence.

Let X be a nonempty convex subset of a topological vector space E and Y a nonempty subset of a topological vector space Z. The correspondence $T : X \to 2^Y$ is said to be weakly naturally quasi-concave (WNQ) ([26]) if, for each $n \in \mathbb{N}^*$ and for each finite set $\{x_1, x_2, ..., x_n\} \subset X$, there exists $y_i \in T(x_i)$, $(i = 1, 2, ..., n)$ and $g \in C^*(\Delta_{n-1})$, such that $\sum\limits_{i=1}^{n} g_i(\lambda_i) y_i \in T(\sum\limits_{i=1}^{n} \lambda_i x_i)$ for every $(\lambda_1, \lambda_2, ..., \lambda_n) \in \Delta_{n-1}$.

We proved in [26] the following fixed point result.

Lemma 2 *([26]) Let Y be a nonempty subset of a topological vector space E, and K be a $(n-1)$- dimensional simplex in E. Let $T : K \to 2^Y$ be an weakly naturally quasi-concave correspondence, and $f : Y \to K$ be a continuous function. Then, there exists $x^* \in K$ such that $x^* \in f \circ T(x^*)$.*

Now, we recall the generalized KKM mappings, firstly introduced by Park [23].

Let X be a convex subset of a linear space, let Y be a topological space and $T, G : X \to 2^Y$ be two correspondences. We call G a generalized KKM mapping w.r.t. T if $T(\mathrm{co}A) \subset G(A)$ for each finite subset A of X. We say that T has the KKM property if G is a generalized KKM mapping w.r.t. T and the family $\{\overline{G}(x) : x \in X\}$ has the finite intersection property. We denote $KKM(X, Y) = \{T : X \to 2^Y : T$ has the KKM property$\}$.

The following lemma is a particular case of Lemma 2.2 in [18].

Lemma 3 *Let X be a topological space, Y be a convex set in a topological vector space. Let $T \in KKM(Y, X)$ be compact and $G : Y \to 2^X$ be a generalized KKM map w.r.t T. Then, $\overline{T(Y)} \cap \bigcap_{y \in Y} \overline{G(y)} \neq \emptyset$.*

Let X be a nonempty convex subset of a topological vector space E, Z be a real topological vector space, Y be a subset of Z and C be a pointed closed convex cone in Z with its interior $\text{int}C \neq \emptyset$. Let $T : X \to 2^Z$ be a correspondence with nonempty values. T is said to be (in the sense of [[16], Definition 3.6]) type-(v) *properly $C-$quasi-convex on X* [13], if for any $x_1, x_2 \in X$ and $\lambda \in [0,1]$, either $T(\lambda x_1 + (1-\lambda)x_2) \subset T(x_1) - C$ or $T(\lambda x_1 + (1-\lambda)x_2) \subset T(x_2) - C$.

3 Coincidence theorems and applications

There are many existed theorems that provide conditions on how to obtain coincidence points for "adequate" correspondences, that is, correspondences which satisfy reasonable assumptions concerning the convexity of the images and continuity. However, there is much less guidance available on how to obtain similar results under constraints regarding these assumptions. This section addresses such a challenge and the possible applications of a new point of view to some classes of generalized vector equilibrium problems and minimax inequalities.

3.1 Coincidence theorems and generalized vector equilibrium problems

In this subsection, we prove some generalized coincidence theorems for the case when the images of the correspondences are not convex. We work with new types of properly quasi-convex-like correspondences. By applying our results, we obtain new theorems concerning the existence of solutions for generalized vector equilibrium problems.

Now, we present the first result of this subsection. By using Lemma 2, we establish Theorem 1.

Theorem 1 *Let X be a simplex in a topological vector space E, let Y be a Hausdorff space and $T, G : X \to 2^Y$ be correspondences satisfying:*

a) T is weakly naturally quasi-concave and compact;
b) for each $y \in \overline{T(X)}$, $G^{-1}(y)$ is convex;
c) $\overline{T(X)} = \cup_{x \in X} IntG(x)$.
Then, there exists $x^* \in X$ such that $T(x^*) \cap G(x^*) \neq \emptyset$.

Proof Since $\overline{T(X)}$ is compact, assumption c) implies that there exists $x_1, x_2, ..., x_n \in X$ such that $\overline{T(X)} \subset \cup_{i=1}^n \text{Int}G(x_i)$. We consider $\{\lambda_1, \lambda_2, ...\lambda_n\}$ the partition of unity corresponding to $\{\text{int}G(x_i)\}_{i=1,...,n}$. We denote $K = \text{co}\{x_1, x_2, ...x_n\} \subset X$ and we define $f : \overline{T(X)} \to K$ by

$f(y) = \sum_{i=1}^n \lambda_i(y) x_i$ for each $y \in \overline{T(X)}$.

We note that $\lambda_i(y) \neq \emptyset$ if only if $y \in \text{int}G(x_i)$ or, $x_i \in G^{-1}(y)$.

The continuity of f is obvious and assumption 2) implies $f(y) \in \text{co}\{x_i : x_i \in G^{-1}(y)\} \subset G^{-1}(y)$ for each $y \in T(X)$.

According to Lemma 2, there exists $x^* \in X$ such that $x^* \in fT(x^*)$. In addition, $f^{-1}(x^*) \subset G(x^*)$ and we obtain that $T(x^*) \cap G(x^*) \neq \emptyset$.

We define the following type of correspondences.

Definition 1 *Let X be a nonempty convex subset of a topological vector space E and Y be a real topological vector space. Let $T, G : X \to 2^Y$ be correspondences with nonempty values. T is said to be properly quasi-convex w.r.t. G on X, if for each $n \in \mathbb{N}$, $n \geq 2$, $x_1, x_2, ..., x_n \subset Y$, $x \in co\{x_1, x_2, ..., x_n\}$, there exists $i_0 \in \{1, 2, ..., n\}$ such that $T(x) \subset G(x_{i_0})$.*

Example 1 Let $S'_+((0,0), x) := \{(u, v) \in [-1,1] \times [0,1] : u^2 + v^2 \leq x^2\}$ and

$S'_-((0,0), x) := \{(u, v) \in [-1,1] \times [-1,0] : u^2 + v^2 \leq x^2\}$.

Let us define $T, G : [0,1] \to 2^{[-1,1]\times[-1,1]}$ by

$T(x) := \begin{cases} S'_+((0,0), x), & \text{if } x \in [0,1],\ x \neq 1/4; \\ S'_-((0,0), x), & \text{if } x = 1/4; \end{cases}$ and

$G(x) := \begin{cases} S'_+((0,0), x) \cup \{(x, x) : x \in [0,1]\}, & \text{if } x \in [0,1],\ x \neq 1/4; \\ S'_-((0,0), x) \cup \{(x, x) : x \in [-1,0]\}, & \text{if } x = 1/4. \end{cases}$

Then, T is properly quasi-convex w.r.t. G.

Remark 1 If T is properly quasi-convex and $T(x) \subseteq G(x)$ for each $x \in X$, then T is properly quasi-convex w.r.t. G.

Example 2 Let us define $T, G : \mathbb{R} \to 2^{\mathbb{R}}$ by

$T(x) := \begin{cases} (-\infty, 4), & \text{if } x \in (-\infty, 2]; \\ [x, 3) & \text{if } x \in (2,3); \\ (2, \infty), & \text{if } x \in [3, \infty); \end{cases}$ and

$G(x) := \begin{cases} (-\infty, 5), & \text{if } x \in (-\infty, 2]; \\ [x, 5] & \text{if } x \in (2,3); \\ (4, \infty), & \text{if } x \in [3, \infty). \end{cases}$

T is properly quasi-convex and $T(x) \subseteq G(x)$ for each $x \in \mathbb{R}$. Then, T is properly quasi-convex w.r.t. G.

Example 3 shows that it is not necessary that all images of the correspondences T to be included in the images of the correspondence G.

Example 3 Let us define $T, G : [2,3] \to 2^{\mathbb{R}}$ by

$T(x) := \begin{cases} (3, \infty), & \text{if } x = 2; \\ [x, 3) & \text{if } x \in (2,3); \\ (-\infty, 5), & \text{if } x = 3; \end{cases}$ and

$G(x) := \begin{cases} (-\infty, 4), & \text{if } x = 2; \\ [x, 3] & \text{if } x \in (2,3); \\ (2, \infty), & \text{if } x = 3. \end{cases}$

Then, T is properly quasi-convex w.r.t. G.

Remark 2 If $T : X \to 2^Y$ is properly quasi-convex w.r.t. $G : X \to 2^Y$, then, G is a generalized KKM map w.r.t T.

In order to show this assertion, we consider $\{x_1, x_2, ..., x_n\} \subset X$. We want to prove that $T(x) \subset \cup_{i=1}^{n} G(x_i)$ for each $x \in$co$\{x_1, x_2, ..., x_n\}$. Suppose, to the contrary, that there exist $y^* \in$co$\{x_1, x_2, ..., x_n\}$ and $y^* \in T(x^*)$ such that $y^* \notin \cup_{i=1}^{n} G(x_i)$. Then, $y^* \notin G(x_i)$ for each $i \in \{1, 2, ..., n\}$ and T is not properly quasi-convex w.r.t. G.

The T-properly quasi-convex sets are introduced below.

Definition 2 *Let X be a topological space, Y be a convex set in a topological vector space. Let $T : Y \to 2^X$ and $A \subseteq X \times Y$. The set A is said to be T-properly quasi-convex if for each $n \in \mathbb{N}$, $n \geq 2$, $y_1, y_2, ..., y_n \in Y$, $y \in co\{y_1, y_2, ..., y_n\}$ and $x \in T(y)$, there exists $i_0 \in \{1, 2, ..., n\}$ such that $(x, y_{i_0}) \in A$.*

Example 4 Let us define $T : [0, 2] \to 2^{[-1,1]}$ by

$$T(y) := \begin{cases} 1, & \text{if} \quad x \in [0, 2],\ x \neq 1; \\ -1, & \text{if} \quad x = 1 \end{cases} \text{ and}$$

$A := \{(0, v) : y \in [0, 1) \cup (1, 2)\} \cup \{(x, x) : x \in [0, 1)\} \cup \{(x, x - 2) : x \in [0, 1)\} \cup \{(-1, 1)\}$.

A is T-properly quasi-convex.

We emphasize the relation between T-properly quasi-convex sets and T-properly quasi-convex correspondences.

Remark 3 We note that if A is T-properly quasi-convex and if we define $G : X \to 2^Y$ by $G(x) = \{y \in Y : (x, y) \in A\}$ for each $x \in X$, then, T is properly quasi-convex w.r.t. G^{-1}.

Example 5 In the above example, $G : X \to 2^Y$ is defined by

$$G(x) = \{y \in Y : (x, y) \in A\} = \begin{cases} [0, 1) \cup (1, 2) & \text{if} \quad x = 1; \\ \{(x, x)\} \cup \{(x, x - 2)\} & \text{if } x \in [0, 1); \\ \{1\} & \text{if} \quad x = -1 \end{cases}$$

and

$G^{-1} : Y \to 2^X$ is defined by

$$G^{-1}(y) = \{x \in Y : (x, y) \in A\} = \begin{cases} \{1\} \cup \{y\} & \text{if} \quad y \in [0, 1); \\ \{1\} \cup \{2 - y\} & \text{if } y \in (1, 2]; \\ \{-1\} & \text{if} \quad y = 1. \end{cases}$$

Then, T is properly quasi-convex w.r.t. G^{-1}.

A generalization of T-properly quasi-convex sets is introduced now.

Definition 3 *Let X be a topological space, Y be a convex set in a topological vector space. Let $T : Y \to 2^X$, $Q : Y \to 2^Y$ and $A \subseteq X \times Y$. The set A is said to be T-properly quasi-convex with respect to Q if, for each $n \in \mathbb{N}$, $n \geq 2$, $y_1, y_2, ..., y_n \in Y$, $y \in co\{y_1, y_2, ..., y_n\}$ and $x \in T(y)$, there exists $i_0 \in \{1, 2, ..., n\}$ such that $(x, z) \in A$ for each $z \in Q(y_{i_0})$.*

We note that if A is T-properly quasi-convex with respect to Q and if we define $G : X \to 2^Y$ by $G(x) = \{y \in Y : P(x) \cap Q(y) = \emptyset\}$, where $P(x) = \{y \in Y : (x, y) \notin A\}$ for each $x \in X$, then, T is properly quasi-convex w.r.t. G^{-1}.

Theorem 2 is a Fan type geometric result involving T-properly quasi-convex sets.

Theorem 2 *Let X be a topological space, Y be a convex set in a topological vector space and let $T \in KKM(Y, X)$ be compact. Let $A \subseteq X \times Y$ satisfying the following conditions:*
a) A is T-properly quasi-convex;
b) the correspondence $G : X \to 2^Y$, defined by $G(x) = \{y \in Y : (x, y) \in A\}$ for each $x \in X$, is such that G^{-1} is transfer closed-valued.
Then, there exists $x^* \in \overline{T(Y)}$ such that $(x^*, y) \in A$ for all $y \in Y$.

Proof According to Remarks 2 and 3, G^{-1} is a generalized KKM map w.r.t T. Hence, we can apply Lemma 3 and we obtain that $\overline{T(Y)} \cap \bigcap_{y \in Y} \overline{G^{-1}(y)} \neq \emptyset$. Consequently, there exists $x^* \in \overline{T(Y)}$ such that $(x^*, y) \in A$ for all $y \in Y$.

We give an example to illustrate the usage of Theorem 2.

Example 6 Let us define $T : [0, 2] \to 2^{[-1,1]}$ by

$T(y) := \begin{cases} 1, & \text{if} \quad x \in [0, 2],\ x \neq 1; \\ -1, & \text{if} \quad x = 1 \end{cases}$ and

$A := \{(0, v) : y \in [0, 2]\} \cup \{(x, x) : x \in [0, 1)\} \cup \{(x, x - 2) : x \in [0, 1)\} \cup \{(-1, 1)\}$.

A is T-properly quasi-convex.

$G^{-1} : [0, 2] \to 2^{[-1,1]}$ is defined by

$$G^{-1}(y) = \{x \in [-1, 1] : (x, y) \in A\} = \begin{cases} \{1\} \cup \{y\} & \text{if} \quad y \in [0, 1); \\ \{1\} \cup \{2 - y\} & \text{if } y \in (1, 2]; \\ \{-1\} \cup \{1\} & \text{if} \quad y = 1. \end{cases}$$

G^{-1} is transfer closed-valued.

Then, there exists $x^* = 1 \in [-1, 1]$ such that $(x^*, y) \in A$ for all $y \in [0, 2]$.

Theorem 3 is a Fan type geometric result involving T-properly quasi-convex sets with respect to Q.

Theorem 3 *Let X be a topological space, Y be a convex set in a topological vector space and let $T \in KKM(Y, X)$ be compact. Let $A \subseteq X \times Y$ satisfying the following conditions:*
a) A is T-properly quasi-convex with respect to $Q : Y \to 2^Y$;
b) the correspondence $G : X \to 2^Y$, defined by $G(x) = \{y \in X : P(x) \cap Q(y) = \emptyset\}$ for each $x \in X$, where $P(x) = \{y \in Y : (x, y) \notin A\}$ for each $x \in X$, is such that G^{-1} is transfer closed-valued.
Then, there exists $x^* \in \overline{T(Y)}$ such that $(x^*, z) \in A$ for all $z \in Q(Y)$.

Proof According to Remarks 2 and 3, G^{-1} is a generalized KKM map w.r.t T. Hence, we can apply Lemma 3 and we obtain that $\overline{T(Y)}\cap\bigcap_{y\in Y}\overline{G^{-1}(y)}\neq\emptyset$. Consequently, there exists $x^*\in\overline{T(Y)}$ such that $(x^*,z)\in A$ for all $z\in Q(Y)$.

Now, we obtain a coincidence-like result, for the case when the convexity of the images of the correspondence P is missing.

Theorem 4 *Let X be a topological space and Y be a convex set in a topological vector space. Let $T\in KKM(Y,X)$ be compact. Let $P:X\to 2^Y$ be a correspondence such that $X=\cup_{y\in Y}IntP^{-1}(y)$. Then, there exist $n\in\mathbb{N}$, $n\geq 2$, $B=\{y_1^*,y_2^*,...,y_n^*\}\subset Y$, $y^*\in co\{y_1^*,y_2^*,...,y_n^*\}$ and $x^*\in T(y^*)$, such that $y_i^*\in P(x^*)$ for each $i\in\{1,2,...,n\}$.*

Proof Suppose, to the contrary, that for each $n\in\mathbb{N}$, $n\geq 2$, $y_1,y_2,...,y_n\subset Y$, $y\in\text{co}\{y_1,y_2,...,y_n\}$ and $x\in T(y)$, there exists $i_0\in\{1,2,...,n\}$ such that $y_{i_0}\notin P(x)$. Then, $(x,y_{i_0})\in A$, where $A=\{(x,y)\in X\times Y:(x,y)\notin\text{Gr}P\}\subseteq X\times Y$. If we define $G:X\to 2^Y$ by $G(x)=\{y\in Y:(x,y)\in A\}=\{y\in Y:y\notin P(x)\}$, we can prove that G^{-1} is transfer closed-valued. In order to do this, we notice that the assumption $X=\cup_{y\in Y}\text{Int}P^{-1}(y)$ and Lemma 1 imply that P^{-1} is transfer open-valued and P is nonempty valued. The relation between the correspondences P and G leads us to the conclusion that G^{-1} is transfer closed-valued. By applying Theorem 2, we obtain that there exists $x^*\in\overline{T(Y)}$ such that $(x^*,y)\in A$ for all $y\in Y$. Consequently, $P(x^*)=\emptyset$, which is a contradiction. Hence, there exist $n\in\mathbb{N}$, $n\geq 2$, $\{y_1^*,y_2^*,...,y_n^*\}\subset Y$, $y^*\in\text{co}\{y_1^*,y_2^*,...,y_n^*\}$ and $x^*\in T(y^*)$, such that $y_i^*\in P(x^*)$ for each $i\in\{1,2,...,n\}$.

Remark 4 If $P(x^*)$ is convex, then, $y^*\in P(x^*)$ and we obtain a coincidence theorem.

We are establishing the following coincidence-like theorem.

Theorem 5 *Let X be a topological space and Y be a convex set in a topological vector space. Let $T\in KKM(Y,X)$ be compact. Let $P:X\to 2^Y$ be a nonempty valued, lower semicontinuous correspondence and let $Q:X\to 2^Y$ be open valud, such that, for each $x\in X$, there exists $y\in Y$ such that $P(x)\cap Q(y)\neq\emptyset$. Then, there exist $n\in\mathbb{N}$, $n\geq 2$, $B=\{y_1^*,y_2^*,...,y_n^*\}\subset Y$, $y^*\in co\{y_1^*,y_2^*,...,y_n^*\}$ and $x^*\in T(y^*)$, such that $Q(y_i^*)\cap P(x^*)\neq\emptyset$ for each $i\in\{1,2,...,n\}$.*

Proof Suppose, to the contrary, that for each $n\in\mathbb{N}$, $n\geq 2$, $y_1,y_2,...,y_n\subset Y$, $y\in\text{co}\{y_1,y_2,...,y_n\}$ and $x\in T(y)$, there exists $i_0\in\{1,2,...,n\}$ such that for each $z\in Q(y_{i_0})$, $z\notin P(x)$. Then, for each $n\in\mathbb{N}$, $n\geq 2$, $y_1,y_2,...,y_n\in Y$, $y\in\text{co}\{y_1,y_2,...,y_n\}$ and $x\in T(y)$, there exists $i_0\in\{1,2,...,n\}$ such that $(x,z)\in A$ for each $z\in Q(y_{i_0})$, where $A=\{(x,y)\in X\times Y:(x,y)\notin\text{Gr}P\}\subseteq X\times Y$.

If we define $G : X \to 2^Y$ by $G(x) = \{y \in Y : P(x) \cap Q(y) = \emptyset\}$ for each $x \in X$, we can prove that G^{-1} is transfer closed-valued. $Y \backslash G^{-1}(y) = \{x \in X : P(x) \cap Q(y) \neq \emptyset\}$ is open. Then, $\underline{G^{-1}(y)}$ is closed. By applying Theorem 3, we obtain that there exists $x^* \in \overline{T(Y)}$ such that $(x^*, y) \in A$ for all $y \in Q(Y)$. Consequently, $P(x^*) \cap Q(Y) = \emptyset$, which is a contradiction. Hence, there exist $n \in \mathbb{N}$, $n \geq 2$, $\{y_1^*, y_2^*, ..., y_n^*\} \subset Y$, $y^* \in$co$\{y_1^*, y_2^*, ..., y_n^*\}$ and $x^* \in T(y^*)$, such that $Q(y_i^*) \cap P(x^*) \neq \emptyset$ for each $i \in \{1, 2, ..., n\}$.

Theorem 4 can be generalized in the following way.

Theorem 6 *Let X be a topological space and Y be a convex set in a topological vector space. Let $T \in KKM(Y, X)$ be compact. Let $P : X \to 2^Y$ and $Q : X \to 2^Y$ be correspondences such that P is nonempty valued and $X = \cup_{y \in Y} Int(G')^{-1}(y)$, where $G' : X \to 2^Y$ is defined by $G'(x) = \{y \in Y : P(x) \cap Q(y) \neq \emptyset\}$ for each $x \in X$. Then, there exist $n \in \mathbb{N}$, $n \geq 2$, $B = \{y_1^*, y_2^*, ..., y_n^*\} \subset Y$, $y^* \in co\{y_1^*, y_2^*, ..., y_n^*\}$ and $x^* \in T(y^*)$, such that $Q(y_i^*) \cap P(x^*) \neq \emptyset$ for each $i \in \{1, 2, ..., n\}$.*

Proof Suppose, to the contrary, that for each $n \in \mathbb{N}$, $n \geq 2$, $y_1, y_2, ..., y_n \subset Y$, $y \in$co$\{y_1, y_2, ..., y_n\}$ and $x \in T(y)$, there exists $i_0 \in \{1, 2, ..., n\}$ such that for each $z \in Q(y_{i_0})$, $z \notin P(x)$. Then, for each $n \in \mathbb{N}$, $n \geq 2$, $y_1, y_2, ..., y_n \in Y$, $y \in$co$\{y_1, y_2, ..., y_n\}$ and $x \in T(y)$, there exists $i_0 \in \{1, 2, ..., n\}$ such that $(x, z) \in A$ for each $z \in Q(y_{i_0})$, where $A = \{(x, y) \in X \times Y : (x, y) \notin \text{Gr}P\} \subseteq X \times Y$.

If we define $G : X \to 2^Y$ by $G(x) = \{y \in Y : P(x) \cap Q(y) = \emptyset\}$, we can prove that G^{-1} is transfer closed-valued. In order to do this, we notice that the assumption $X = \cup_{y \in Y} \text{Int}G'^{-1}(y)$ and Lemma 1 imply that $(G')^{-1}$ is transfer open-valued. The relation between the correspondences G' and G leads us to the conclusion that G^{-1} is transfer closed-valued. By applying Theorem 2, we obtain that there exists $x^* \in \overline{T(Y)}$ such that $(x^*, y) \in A$ for all $y \in Y$. Consequently, $P(x^*) = \emptyset$, which is a contradiction. Hence, there exist $n \in \mathbb{N}$, $n \geq 2$, $\{y_1^*, y_2^*, ..., y_n^*\} \subset Y$, $y^* \in$co$\{y_1^*, y_2^*, ..., y_n^*\}$ and $x^* \in T(y^*)$, such that $Q(y_i^*) \cap P(x^*) \neq \emptyset$ for each $i \in \{1, 2, ..., n\}$.

An application of Theorem 4 is provided in order to establish an existence result for solutions of a generalized vector equilibrium problem.

We consider the following generalized vector equilibrium problem:

Let X be a topological space, Y be a convex set in a topological vector space and let Z be a topological vector space. We consider a correspondence $C : X \to 2^Z$ such that, for each $x \in X$, $C(x)$ is a closed and convex cone with int$C(x) \neq \emptyset$. Let $F : X \times Y \to 2^Z \backslash \{\emptyset\}$.

Find $x^* \in X$ such that $F(x^*, y) \nsubseteq -\text{int}C(x^*)$ for each $y \in Y$.

The correspondence P will be needed in our proof. Let $P : X \to 2^Y$ be defined by $P(x) = \{y \in Y : F(x, y) \subseteq -\text{int}C(x)\}$ for each $x \in X$.

We are ready to prove Theorem 7.

Theorem 7 *Let X be a topological space, Y be a convex set in a topological vector space and let Z be a topological vector space. Let $T \in KKM(Y, X)$*

such that, for each compact subset A of Y, $\overline{T(A)}$ is compact. Let $F : X \times Y \to 2^Z \backslash \{\emptyset\}$ and $C : X \to 2^Z$ such that, for each $x \in X$, $C(x)$ is a pointed, closed and convex cone with int$C(x) \neq \emptyset$. Assume that:

a) T is properly quasi-convex w.r.t. G, where $G : X \to 2^Y$ is defined by $G(x) = \{y \in Y : F(x, y) \nsubseteq -\mathit{int}C(x)\}$ for each $x \in X$;
b) $X = \cup_{y\in Y} \mathit{Int}P^{-1}(y)$.

Then, there exists $x^* \in X$ such that $F(x^*, y) \nsubseteq -\text{int}C(x^*)$ for each $y \in Y$.

Proof Suppose, to the contrary, that the considered equilibrium problem does not have any solutions. It follows that the correspondence P has nonempty values. According to Theorem 4, there exist $n \in \mathbb{N}$, $n \geq 2$, $\{y_1^*, y_2^*, ..., y_n^*\} \subset Y$, $y^* \in$co$\{y_1^*, y_2^*, ..., y_n^*\}$ and $x^* \in T(y^*)$, such that $y_i^* \in P(x^*)$ for each $i \in \{1, 2, ..., n\}$. Then, $y_i^* \notin G(x^*)$ for each $i \in \{1, 2, ..., n\}$ and, consequently, T is not properly quasi-convex w.r.t. G. This fact contradicts a).

Theorem 8 *Let X be a topological space, Y be a convex set in a topological vector space and let Z be a topological vector space. Let $T \in KKM(Y, X)$ such that, for each compact subset A of Y, $\overline{T(A)}$ is compact. Let $F : X \times Y \to 2^Z \backslash \{\emptyset\}$ and $C : X \to 2^Z$ such that, for each $x \in X$, $C(x)$ is a pointed, closed and convex cone with int$C(x) \neq \emptyset$. Assume that:*

a) T is properly quasi-convex w.r.t. G, where $G : X \to 2^Y$ is defined by $G(x) = \{y \in Y : F(x, y) \nsubseteq -\mathit{int}C(x)\}$ for each $x \in X$;
b) for each $x \in X$, $F(\cdot, y) : x \to 2^Z \backslash \{\emptyset\}$ is u.s.c. with nonempty compact values and the map $W : X \to 2^Z$ defined by $W(x) = Z \backslash$(-int$C(x))$ is u.s.c;
c) for each $x \in X$, there exists $y \in Y$ such that $F(x, y) \subseteq -\mathit{int}C(x)$.

Then, there exists $x^* \in X$ such that $F(x^*, y) \nsubseteq -\text{int}C(x^*)$ for each $y \in Y$.

Proof We will prove that for each $y \in Y$, $P^{-1}(y)$ is open. In order to prove this, we consider $x \in \overline{X \backslash P^{-1}(y)}$ and a net $\{x_\alpha\}_{\alpha\in\Lambda}$ in $X \backslash P^{-1}(y)$ such that $x_\alpha \to x$. Since $x_\alpha \in X \backslash P^{-1}(y)$ for each $\alpha \in \Lambda$, we have that $F(x_\alpha, y) \nsubseteq -\text{int}C(x_\alpha)$. Then, for each for each $\alpha \in \Lambda$, there exists $z_\alpha \in F(x_\alpha, y)$ such that $z_\alpha \in Z \backslash$(-int$C(x_\alpha))$. Assumption b) implies that $z \in F(x, y)$ and $z \in Z \backslash (-\text{int}C(x_\alpha))$, that is, $F(x, y) \nsubseteq -\text{int}C(x)$. Consequently, $x \in X \backslash P^{-1}(y)$. This shows that $X \backslash P^{-1}(y)$ is closed and $P^{-1}(y)$ is open for each $y \in Y$.

Assumption c) implies that for each $x \in X$, $P(x)$ is nonempty. According to Lemma 1, $X = \cup_{y\in Y} \text{Int}P^{-1}(y)$ and we can use Theorem 7 in order to obtain the conclusion.

Now, by applying Lemma 3, we derive the following coincidence-like theorem.

Theorem 9 *Let X be a topological space, Y be a convex set in a topological vector space and Z be a nonempty set.*

Let $P : X \to 2^Z$, $Q : Y \to 2^Z$ *and* $T : Y \to 2^X$ *be correspondences satisfying the following assumptions*:

a) there exists $y_0 \in Y$ *such that* $P(x) \cap Q(y_0) \neq \emptyset$ *for each* $x \in X$;

b) $T \in KKM(Y, X)$ *is compact.*

Then, there exist $n \in \mathbb{N}, n \geq 2$, $B = \{y_1^*, y_2^*, ..., y_n^*\} \subset Y$, $y^* \in$co$\{y_1^*, y_2^*, ..., y_n^*\}$ *and* $x^* \in T(y^*)$, *such that* $P(x^*) \cap Q(y_i^*) \neq \emptyset$ *for each* $i \in \{1, 2, ..., n\}$.

Proof Suppose, to the contrary, that for each $n \in \mathbb{N}$, $n \geq 2$, $y_1, y_2, ..., y_n \subset Y$, $y \in$co$\{y_1, y_2, ..., y_n\}$ and $x \in T(y)$, there exists $i_0 \in \{1, 2, ..., n\}$ such that $Q(y_{i_0}) \cap P(x) = \emptyset$. Then, for each $y_1, y_2, ..., y_n \subset Y$, $y \in$co$\{y_1, y_2, ..., y_n\}$, there exists $i_0 \in \{1, 2, ..., n\}$ such that $T(y) \subset G(y_{i_0})$, where $G : Y \to 2^X$ is defined by $G(y) = \{x \in X : P(x) \cap Q(y) = \emptyset\}$ for each $y \in Y$. We note that T is properly quasi-convex w.r.t. G.

We claim that G is generalized KKM w.r.t. T. In order to prove this, let us suppose, to the contrary, that there exist $B = \{y_1^*, y_2^*, ..., y_n^*\}$ and $x^* \in T(\mathrm{co}B) \setminus \bigcup_{y^* \in B} G(y^*)$. Then, there exists $y^* \in$co$B$ *and* $x^* \in T(y^*)$. Since T is properly quasi-convex w.r.t. G, we have that there exists $i_0 \in \{1, 2, ..., n\}$ such that $T(y^*) \subset G(y_{i_0}^*)$, that is, $P(x^*) \cap Q(y_{i_0}^*) = \emptyset$. Further, we have that $x^* \notin \bigcup_{i=1}^n G(y_i^*)$. Therefore, $P(x^*) \cap Q(y_i^*) \neq \emptyset$ for each $i \in \{1, 2, ..., n\}$, which contradicts the last assertion. Consequently, G is generalized KKM w.r.t. T.

According to Lemma 3, there exists $x^* \in \bigcap_{y \in Y} \overline{G(y)}$. Obviously, $x^* \in \overline{G(y)}$ for each $y \in Y$ and then, for each $y \in Y$, there exists a neighborhood V^y of x^* such that $V^y \cap G(y) \neq \emptyset$. It follows that for each $y \in Y$, there exists x^y such that $x^y \in G(y)$, that is, $P(x^y) \cap Q(y) = \emptyset$, which contradicts assumption a).

Theorem 10 is a consequence of Theorem 9. It generalizes some results concerning the existence of the maximal elements.

Theorem 10 *Let* X *be a topological space,* Y *be a convex set in a topological vector space and* Z *be a nonempty set.*

Let $P : X \to 2^Z$, $Q : Y \to 2^Z$ *and* $T : Y \to 2^X$ *be correspondences satisfying the following assumptions*:

a) T *is properly quasi-convex w.r.t.* G, *where* $G : Y \to 2^X$ *is defined by* $G(y) = \{x \in X : P(x) \cap Q(y) = \emptyset\}$ for each $y \in Y$.

b) $T \in KKM(Y, X)$ *is compact.*

Then, for each $y \in Y$, *there exists* x_0^y *such that* $P(x_0^y) \cap Q(y) = \emptyset$.

Proof Suppose, to the contrary, that the conclusion of the theorem does not hold. Therefore, there exists $y_0 \in Y$ such that $P(x) \cap Q(y_0) \neq \emptyset$ for each $x \in X$ and then, assumption a) of Theorem 9 is fulfilled. By applying Theorem 9, we obtain that there exist $n \in \mathbb{N}$, $n \geq 2$, $B = \{y_1^*, y_2^*, ..., y_n^*\} \subset Y$, $y \in$co$B$ and $x^* \in T(y^*)$, such that $P(x^*) \cap Q(y_i^*) \neq \emptyset$ for each $i \in \{1, 2, ..., n\}$, which contradicts a).

Remark 5 If $Q(y) = Z$ for each $y \in Y$, then Theorem 10 asserts the existence of the maximal elements of the correspondence T.

Now, we consider the following generalized vector equilibrium problem:

Let X be a topological space, Y be a convex set in a topological vector space and Z be a topological vector space.

Let $F : X \times Z \to 2^Z \backslash \{\emptyset\}$, $Q : Y \to 2^Z$ and $C : X \to 2^Z$ such that, for each $x \in X$, $C(x)$ is a pointed, closed and convex cone with $\text{int}C(x) \neq \emptyset$. Let $F : X \times Z \to 2^Z \backslash \{\emptyset\}$, $Q : Y \to 2^Z$ and $C : X \to 2^Z$ such that, for each $x \in X$, $C(x)$ is a pointed, closed and convex cone with $\text{int}C(x) \neq \emptyset$.

Then, for each $y \in Y$, find $x^* \in X$ such that $F(x^*, z) \nsubseteq -\text{int}C(x^*)$ for each $z \in Q(y)$.

The correspondence P will be needed in our proof. We define $P : X \to 2^Z$ by $P(x) = \{z \in Z : F(x, z) \subseteq -\text{int}C(x)\}$ for each $x \in X$.

Theorem 11 concerns the existence of solutions for the above generalized vector equilibrium problem.

Theorem 11 *Let X be a topological space, Y be a convex set in a topological vector space and Z be a topological vector space. Let $T \in KKM(Y, X)$ such that, for each compact subset A of Y, $\overline{T(A)}$ is compact. Let $F : X \times Z \to 2^Z \backslash \{\emptyset\}$, $Q : Y \to 2^Z$ and $C : X \to 2^Z$ such that, for each $x \in X$, $C(x)$ is a pointed, closed and convex cone with int$C(x) \neq \emptyset$. Assume that T is properly quasi-convex w.r.t. G, where $G : Y \to 2^X$ is defined by $G(y) = \{x \in X : P(x) \cap Q(y) = \emptyset\}$ for each $y \in Y$.*

Then, for each $y \in Y$, there exists $x^* \in X$ such that $F(x^*, z) \nsubseteq -intC(x^*)$ for each $z \in Q(y)$.

Proof According to Theorem 10, for each $y \in Y$, there exists x^* such that $P(x^*) \cap Q(y) = \emptyset$, that is, for each $y \in Y$, there exists x^* such that $R(x^*, y) = \{z \in Q(y) : F(x^*, z) \subseteq -\text{int}C(x^*)\} = \emptyset$.

Consequently, for each $y \in Y$, there exists $x^* \in X$ such that $F(x^*, z) \nsubseteq -\text{int}C(x^*)$ for each $z \in Q(y)$.

A new coincidence-like theorem can be proved under new assumptions.

Theorem 12 *Let X be a topological space, Y be a convex set in a topological vector space and Z be a nonempty set.*

Let $P : X \to 2^Z$, $Q : Y \to 2^Z$ and $T : Y \to 2^X$ be correspondences satisfying the following assumptions:

a) for each $x \in X$, there exists $y \in Y$ such that $P(x) \cap Q(y) \neq \emptyset$;

b) for each $x \in X$ and $y \in Y$ such that *$P(x) \cap Q(y) \neq \emptyset$, there xists $y' \in Y$ and a neighborhood U of x such that $P(x') \cap Q(y') \neq \emptyset$ for each $x' \in U$;*

c) $T \in KKM(Y, X)$ is compact.

Then, there exist $n \in \mathbb{N}$, $n \geq 2$, $B = \{y_1^*, y_2^*, ..., y_n^*\} \subset Y$, $y^* \in$co$B$ and $x^* \in T(y^*)$, such that $P(x^*) \cap Q(y_i^*) \neq \emptyset$ for each $i \in \{1, 2, ..., n\}$.

Proof Suppose, to the contrary, that for each $n \in \mathbb{N}$, $n \geq 2$, $y_1, y_2, ..., y_n \subset Y$, $y \in$co$\{y_1, y_2, ..., y_n\}$ and $x \in T(y)$, there exists $i_0 \in \{1, 2, ..., n\}$ such that $Q(y_{i_0}) \cap P(x) = \emptyset$. Then, for each $y_1, y_2, ..., y_n \subset Y$, $y \in$co$\{y_1, y_2, ..., y_n\}$, there exists $i_0 \in \{1, 2, ..., n\}$ such that $T(y) \subset G(y_{i_0})$, where $G : Y \to 2^X$ is defined by $G(y) = \{x \in X : P(x) \cap Q(y) = \emptyset\}$ for each $y \in Y$. We note that T is properly quasi-convex w.r.t. G.

We claim that G is generalized KKM w.r.t. T. In order to prove this, let us suppose, to the contrary, that there exist $n \in \mathbb{N}$, $n \geq 2$, $B = \{y_1^*, y_2^*, ..., y_n^*\}$ and $x^* \in T(\text{co}B) \setminus \bigcup_{y^* \in B} G(y^*)$. Then, there exists $y^* \in$co$B$ *and* $x^* \in T(y^*)$. Since T is properly quasi-convex w.r.t. G, we have that there exists $i_0 \in \{1, 2, ..., n\}$ such that $T(y^*) \subset G(y_{i_0}^*)$, that is, $P(x^*) \cap Q(y_{i_0}^*) = \emptyset$. Further, we have that $x^* \notin \bigcup_{i=1}^n G(y_i^*)$. Therefore, $P(x^*) \cap Q(y_i^*) \neq \emptyset$ for each $i \in \{1, 2, ..., n\}$, which contradicts the last assertion. Consequently, G is generalized KKM w.r.t. T.

According to Lemma 3, there exists $x^* \in \bigcap_{y \in Y} \overline{G(y)}$. Assumption b) implies that G is transfer closed-valued and then, $x^* \in \bigcap_{y \in Y} G(y)$. It follows that $P(x^*) \cap Q(y) = \emptyset$ for each $y \in Y$, which contradicts a).

Theorem 13 is a consequence of Theorem 12. It generalizes some results concerning the existence of the maximal elements.

Theorem 13 *Let X be a topological space, Y be a convex set in a topological vector space and Z be a nonempty set.*

Let $P : X \to 2^Z$, $Q : Y \to 2^Z$ and $T : Y \to 2^X$ be correspondences satisfying the following assumptions:

a) T is properly quasi-convex w.r.t. G, where $G : Y \to 2^X$ is defined by $G(y) = \{x \in X : P(x) \cap Q(y) = \emptyset\}$ for each $y \in Y$.

b) $T \in KKM(Y, X)$ is compact.

Then, there exists x_0 such that $P(x_0) \cap Q(y) = \emptyset$ for each $y \in Y$.

Proof Suppose, to the contrary, that the conclusion of the theorem does not hold. Therefore, for each $x \in X$, there exists $y_0 \in Y$ such that $P(x) \cap Q(y_0) \neq \emptyset$ and then, assumption a) of Theorem 12 is fulfilled. By applying Theorem 12, we obtain that there exist $n \in \mathbb{N}$, $n \geq 2$, $B = \{y_1^*, y_2^*, ..., y_n^*\} \subset Y$, $y^* \in$co$B$ and $x^* \in T(y^*)$, such that $P(x^*) \cap Q(y_i^*) \neq \emptyset$ for each $i \in \{1, 2, ..., n\}$, which contradicts a).

Remark 6 If $Q(y) = Z$ for each $y \in Y$, then Theorem 13 asserts the existence of the maximal elements of the correspondence T.

3.2 A ***minimax inequality***

The classical Ky Fan's minimax inequalities [9]-[11] have played an important role in the study of modern nonlinear analysis. This subsection is devoted to the research of a generalized Ky Fan minimax inequality for vector-valued functions.

First, we prove the following interesting theorem.

Theorem 14 *Let X be a topological space, Y and Z be convex sets in topological spaces, $p : X \times Z \to \mathbb{R}$, $q : X \times Z \to \mathbb{R}$, $q : X \times Y \to \mathbb{R}$ functions and α, β, λ real numbers. Suppose that the following conditions are fulfilled:*

a) p is transfer upper semicontinuous in x;

b) for each $x \in X$, $y_1, y_2, ..., y_n \in Y$, $y \in$co$\{y_1, y_2, ..., y_n\}$ if $t(x, y) \geq \lambda$, then there exists $i_0 \in \{1, 2, ..., n\}$ such that $t(x, y_{i_0}) \geq \lambda$;

c) for each $x \in X$, $y \in Y$ and $z \in Z$, if $t(x, y) \geq \lambda$, then $p(x, z) < \alpha$ and $q(y, z) > \beta$;

d) for each $x \in X$ and $z \in Z$ such that $p(x, z) < \alpha$, there exists $y \in Y$ such that $q(y, z) < \beta$;

e) there exists a compact subset K of X such that for each $x \in X \backslash K$, $t(x, y) < \lambda$ for all $y \in Y$;

f) the map $T : Y \to 2^X$, defined by $T(y) = \{x \in X : t(x, y) \geq \lambda\}$ for each $y \in Y$, has the KKM property.

Then, for each $y \in Y$, there exists $x_0 \in X$ such that for $z \in Z$ with the property that $q(y, z) < \beta$, it is true that $p(x_0, z) \geq \alpha$.

Proof We start the proof by defining the correspondences $P : X \to 2^Z$, $Q : Y \to 2^Z$ and $T : Y \to 2^X$ by

$P(x) = \{z \in Z : p(x, z) < \alpha\}$, $Q(y) = \{z \in Z : q(y, z) < \beta\}$ and $T(y) = \{x \in X : t(x, y) \geq \lambda\}$ for each $x \in X$, respectively $y \in Y$.

According to assumption e), $T(Y) \subset K$. It follows that T is a compact correspondence. In addition, according to assumption d), T has the KKM property.

We claim that T is properly quasi-convex w.r.t. G, where $G : Y \to 2^X$ is defined by $G(y) = \{x \in X : P(x) \cap Q(y) = \emptyset\}$ for each $y \in Y$.

In order to prove this, let us consider $y_1, y_2, ..., y_n \in Y$, $y \in$co$\{y_1, y_2, ..., y_n\}$ and $x \in T(y)$. Then, $t(x, y) \geq \lambda$.

Assumption b) implies that there exists $i_0 \in \{1, 2, ..., n\}$ such that $t(x, y_{i_0}) \geq \lambda$. Further, assumption c) holds and we conclude that for each $z \in Z$, $p(x, z) < \alpha$ and $q(y_{i_0}, z) > \beta$, that is, $P(x) \cap Q(y_{i_0}) = \emptyset$. Therefore, $x \in G(y_{i_0})$, which implies that there exists $i_0 \in \{1, 2, ..., n\}$ such that $T(x_\lambda) \subset G(y_{i_0})$. We proved that T is properly quasi-convex w.r.t. G.

Based on assumptions a) and d), we conclude that G is transfer closed-valued.

All assumptions of Theorem 13 are fulfilled. By applying this result, we obtain that there exists $x_0 \in X$ such that $P(x_0) \cap Q(y) = \emptyset$ for each $y \in Y$. Consequently, there exists $x_0 \in X$ such that for each $y \in Y$ and $z^y \in Z$ with the property that $q(y, z^y) < \beta$, it is true that $p(x_0, z^y) \geq \alpha$.

Theorem 15 is a consequence of Theorem 14.

Theorem 15 *Let X be a topological space, Y and Z be convex sets in topological spaces and let $p : X \times Z \to \mathbb{R}_+$, $q : X \times Z \to \mathbb{R}_+$, $q : X \times Y \to \mathbb{R}_+$ be functions. Suppose that the following conditions are fulfilled:*

a) p is transfer upper semicontinuous in x;

b) for each $x \in X$, $y_1, y_2, ..., y_n \in Y$, $y \in$co$\{y_1, y_2, ..., y_n\}$, if $t(x, y) \geq \lambda$, then there exists $i_0 \in \{1, 2, ..., n\}$ such that $t(x, y_{i_0}) \geq \lambda$;

c) for each $x \in X$ and $z \in Z$ such that $p(x, z) < \alpha$, there exists $y \in Y$ such that $q(y, z) < \beta$;

d) for each $x \in X$, $y \in Y$ and $z \in Z$, $t(x, y) \leq \frac{q(y,z)}{p(x,z)}$;

e) for each $\lambda < inf_{x\in X} sup_{y\in Y} t(x, y)$, the map $T : Y \to 2^X$, defined by $T(y) = \{x \in X : t(x, y) \geq \lambda\}$ for each $y \in Y$, has the KKM property.

Then, $inf_{x\in X} sup_{y\in Y} t(x, y) \leq \frac{\sup_{y\in Y} \inf_{z\in Z} q(y,z)}{\inf_{z\in Z} \sup_{x\in X} p(x,z)}$.

Proof Let us suppose that the conclusion does not hold. Hence, $\inf_{x\in X} \sup_{y\in Y} t(x, y) > \frac{\sup_{y\in Y} \inf_{z\in Z} q(y,z)}{\inf_{z\in Z} \sup_{x\in X} p(x,z)}$.

The constants $\alpha, \beta, \lambda \in \mathbb{R}_+$ can be chosen such that $\inf_{x\in X} \sup_{y\in Y} t(x, y) > \lambda$, $\sup_{y\in Y} \inf_{z\in Z} q(y, z) < \beta$, $\inf_{z\in Z} \sup_{x\in X} p(x, z) > \alpha$ and $\lambda > \frac{\beta}{\alpha}$.

We claim that condition c) of the above theorem is fulfilled. Indeed, let $x \in X$, $y \in Y$ such that $t(x, y) \geq \lambda > \frac{\beta}{\alpha}$. According to assumption d) of Theorem 14, $\frac{q(y,z)}{p(x,z)} > \frac{\beta}{\alpha}$ for each $z \in Z$. Then, there exists $\varepsilon > 0$ such that $p(x, z) < \alpha\varepsilon$ and $q(y, z) > \beta\varepsilon$ for each $z \in Z$.

All assumptions of Theorem 14 are fulfilled. By applying this result, we obtain that there exists $x_0 \in X$ such that for each $y \in Y$ and for $z^y \in Z$ with the property that $q(y, z^y) < \beta\varepsilon$, it is true that $p(x_0, z^y) \geq \alpha\varepsilon$.

Therefore, there exists $x_0 \in X$ such that for each $y \in Y$ there exists z^y with the property that $t(x_0, y) < \frac{q(y,z^y)}{p(x_0^y,z^y)} < \frac{\beta\varepsilon}{\alpha\varepsilon} = \frac{\beta}{\alpha} < \lambda$. It follows that $\inf_{x\in X} \sup_{y\in Y} t(x, y) \leq \lambda$, which contradicts the choice of λ.

4 Applications of the KKM principle to general vector variational inclusion problems involving correspondences

The vector variational inclusion problem is considered to be a model which unifies several other problems, for instance, vector variational inequalities, vector optimization problems, equilibrium problems or fixed points theorems. For further information on this topic, the reader is referred to the following list of selected publications: [1], [3]-[8], [12], [15], [20], [28], [29], [33], [34].

We report new results concerning the existence of solutions for the general vector variational inclusion problems under new assumptions. The methodology of the proofs relies on the applications of the KKM principle.

In this subsection, we will use a particular form of the KKM principle. We start by presenting it here. We note that its open version is due to Kim [15] and Shih and Tan [30].

Let X be a subset of a topological vector space and D a nonempty subset of X such that co$D \subset X$.

$T : D \to 2^X$ is called a *KKM correspondence* if $\mathrm{co}N \subset T(N)$ for each $N \in \langle D\rangle$, where $\langle D\rangle$ denotes the class of all nonempty finite subsets of D.

KKM principle Let D be a set of vertices of a simplex S and $T : D \to 2^S$ a correspondence with closed (respectively open) values such that $\mathrm{co}N \subset T(N)$ for each $N \subset D$.
Then, $\bigcap_{z\in D} T(z) \neq \emptyset$.

As application of the KKM principle, we recall the following lemma.

Lemma 4 *Let X be a subset of a topological vector space, D a nonempty subset of X such that $coD \subset X$ and $T : D \to 2^X$ a KKM correspondence with closed (respectively open) values. Then $\{T(z)\}_{z\in D}$ has the finite intersection property.*

4.1 Main results

In this subsection, we study the existence of solutions for the following types of variational relation problems. We emphasize that the method of application of the KKM property is new.

Let X be a nonempty subset of a topological vector space E and let Z be a topological vector space. Let $S_1, S_2 : X \to 2^X$, $T : X\times X \to 2^X$, $F : T(X\times X)\times X\times X \to 2^Z$ and $G : T(X\times X)\times X\times X \to 2^Z$ be correspondences with nonempty values. We consider the following generalized vector problems:
(IP 1): Find $x^* \in S_1(x^*)$ such that $\forall y \in S_2(x^*), \forall t^* \in T(x^*, y), F(t^*, y, x^*) \subseteq G(t^*, x^*, x^*)$ and
(IP 2): Find $x^* \in S_1(x^*)$ such that $\forall y \in S_2(x^*), \forall t^* \in T(x^*, y), F(t^*, y, x^*)\cap G(t^*, x^*, x^*) \neq \emptyset$..

For a motivation and special cases of these considered problems, the reader is referred to [14]. We note that this problem also generalizes the vector equilibrium problems considered in Subsection 3.1

Now, we are establishing an existence theorem for the general vector variational inclusion problem IP 1, by using Lemma 4.

Theorem 16 *Let Z be a Hausdorff topological vector space and let X be a nonempty compact convex subset of a topological vector space E. Let $S_1, S_2 : X \to 2^X$, $T : X \times X \to 2^X$, $F : T(X \times X) \times X \times X \to 2^Z$ and $G : T(X\times X)\times X\times X \to 2^Z$ be correspondences with nonempty values. Assume that:*

i) S_1^{-1} and S_2^{-1} are open-valued and for each $u \in X$, the set $\{x \in X : \exists t \in T(x, u), F(t, u, x) \nsubseteq G(t, x, x)\}$ is open;
ii) S_1 and S_2 are convex valued and for each $x \in X$, the set $\{u \in X : \exists t \in T(x, u), F(t, u, x) \nsubseteq G(t, x, x)\}$ is convex;
iii) the set $A = \{x \in X :$ there exist $u \in S_2(x)$ and $t \in T(x, u)$ such that $F(t, u, x) \nsubseteq G(t, x, x)\}$ is closed;

iv) $\{x \in X : x \in S_2(x)\} = \emptyset$;

v) there exists $M \in \langle A \rangle$ *such that* $\bigcup_{u \in M}[\{x \in X : \exists t \in T(x,u), F(t,u,x) \nsubseteq G(t,x,x)\} \cap S_2^{-1}(u) = X$ or $\bigcup_{u \in M}[(X \setminus A) \cap \ S_1^{-1}(u)] = X$.

Then, there exists $x^* \in S_1(x^*)$ *such that* $\forall y \in S_2(x^*)$, $\forall t^* \in T(x^*, y)$, $F(t^*, y, x^*) \subseteq G(t^*, x^*, x^*)$.

Proof. Let $P, G : X \to 2^X$ be defined by

$P(x) = \{u \in X : \exists t \in T(x,u), F(t,u,x) \nsubseteq G(t,x,x)\}$, for each $x \in X$ and

$G(x) = S_2(x) \cap P(x)$, for each $x \in X$.

We are going to show that there exists $x^* \in X$ such that $x^* \in S_1(x^*)$ and $S_2(x^*) \cap P(x^*) = \emptyset$.

We consider two cases.

Case I.

$A = \{x \in X : P(x) \cap S_2(x) \neq \emptyset\} = \{x \in X : G(x) \neq \emptyset\}$ is nonempty.

The correspondence $G : A \to 2^X$, defined by $G(x) = S_2(x) \cap P(x)$ for each $x \in A$, is nonempty valued on A. It is obvious that for each $u \in X$, $G^{-1}(u) = P^{-1}(u) \cap S_2^{-1}(u)$ is a convex set as intersection of convex sets.

Further, let us define the correspondence $H : X \to 2^X$ by

$H(x) = \begin{cases} G(x), & \text{if } x \in A; \\ S_1(x), & \text{otherwise} \end{cases} = \begin{cases} S_2(x) \cap P(x), & \text{if } x \in A; \\ S_1(x), & \text{otherwise.} \end{cases}$

According to ii), H is convex valued.

For each $u \in X$,

$H^{-1}(u) = \{x \in X : u \in H(x)\} =$

$= \{x \in A : u \in S_2(x) \cap P(x)\} \cup \{x \in X \setminus A : u \in S_1(x)\} =$

$= [P^{-1}(u) \cap S_2^{-1}(u)] \cup [(X \setminus A) \cap \ S_1^{-1}(u)]$.

According to i) and iii), H^{-1} is open-valued.

If v) is satisfied, then, there exists $M \in \langle X \rangle$ such that $\bigcup_{x \in M} H^{-1}(x) = X$.

Let us define $Q : X \to 2^X$ by $Q(x) := X \setminus H^{-1}(x)$ for each $x \in X$.

The correspondence Q is closed-valued and $\bigcap_{x \in M} Q(x) = X \setminus \bigcup_{x \in M} H^{-1}(x) = \emptyset$.

According to Lemma 4, we conclude that Q is not a KKM correspondence. Thus, there exists $N \in \langle X \rangle$ such that co$N \subsetneq Q(N) = \bigcup_{x \in N}(X \setminus H^{-1}(x))$.

Hence, there exists $x^* \in$co$N$ with the property that $x^* \in H^{-1}(x)$ for each $x \in N$, which implies $N \subset H(x^*)$. It is clear that co$N \subset$co$H(x^*) = H(x^*)$. Consequently, $x^* \in$co$N \subset$co$H(x^*) = H(x^*)$, which means that $x^* \in H(x^*)$, that is, x^* is a fixed point for H.

We notice that, if $x^* \in A$, then, $x^* \in S_2(x^*) \cap P(x^*)$, which contradicts iv). Therefore, $x^* \in X \setminus A$ and $x^* \in S_1(x^*)$. Since $x^* \in X \setminus A$, we conclude that $S_2(x^*) \cap P(x^*) = \emptyset$. This shows that $\forall y \in S_2(x^*)$, $\forall t^* \in T(x^*, y)$, $F(t^*, y, x^*) \subseteq G(t^*, x^*, x^*)$.

Case II.

$A = \{x \in X : P(x) \cap S_2(x) \neq \emptyset\} = \{x \in X : G(x) \neq \emptyset\} = \emptyset$.

In this case, $G(x) = \emptyset$ for each $x \in X$.

Let us define $Q : X \to 2^X$ by $Q(x) := X \backslash S_1(x)$ for each $x \in X$. The proof follows the same line as above and we obtain that there exists $x^* \in X$ such that $x^* \in S_1(x^*)$.

Obviously, $G(x^*) = \emptyset$. Consequently, the conclusion holds in Case II.

Remark 7 Assumption i) can be replaced with

i') S_1^{-1} and S_2^{-1} are closed-valued and for each $u \in X$, the set $\{x \in X : \exists t \in T(x,u), Q(t,u,x) \nsubseteq G(t,x,x)\}$ is closed.

In this case, Q is open-valued and the open version of Lemma 4 can be applied.

Using a similar argument as in the proof of the above result, we obtain the following theorem.

Theorem 17 *Let Z be a Hausdorff topological vector space and let X be a nonempty compact convex subset of a topological vector space E. Let $S_1, S_2 : X \to 2^X$, $T : X \times X \to 2^X$, $Q : T(X \times X) \times X \times X \to 2^Z$ and $G : T(X \times X) \times X \times X \to 2^Z$ be correspondences with nonempty values. Assume that:*

i) S_1^{-1} and S_2^{-1} are open-valued and for each $u \in X$, the set $\{x \in X : \exists t \in T(x,u), F(t,u,x) \cap G(t,x,x) = \emptyset\}$ is open;

ii) S_1 and S_2 are convex valued and for each $x \in X$, the set $\{u \in X : \exists t \in T(x,u), F(t,u,x) \cap G(t,x,x) = \emptyset\}$ is convex;

iii) the set $A = \{x \in X :$ there exist $u \in S_2(x)$ and $t \in T(x,u)$ such that $F(t,u,x) \cap G(t,x,x) = \emptyset\}$ is closed;

iv) $\{x \in X : x \in S_2(x)\} = \emptyset$;

v) there exists $M \in \langle A \rangle$ such that $\bigcup_{u \in M}[\{x \in X : \exists t \in T(x,u), F(t,u,x) \cap G(t,x,x) = \emptyset\} = X$ or $\bigcup_{u \in M}[(X \backslash A) \cap \ S_1^{-1}(u)] = X$.

Then, there exists $x^* \in S_1(x^*)$ such that $\forall y \in S_2(x^*)$, $\forall t^* \in T(x^*,y)$, $F(t^*,y,x^*) \cap G(t^*,x^*,x^*) \neq \emptyset$.

Proof. Let $P, G : X \to 2^X$ be defined by

$P(x) = \{u \in X : \exists t \in T(x,u), F(t,u,x) \cap G(t,x,x) = \emptyset\}$, for each $x \in X$ and

$G(x) = S_2(x) \cap P(x)$, for each $x \in X$.

We are going to show that there exists $x^* \in X$ such that $x^* \in S_1(x^*)$ and $S_2(x^*) \cap P(x^*) = \emptyset$.

The rest of the proof follows the same line as the proof of Theorem 16.

We obtain a new existence theorem of solutions for a generalized vector variational inclusion problem.

Theorem 18 *Let Z be a Hausdorff topological vector space and let X be a nonempty compact convex subset of a topological vector space E. Let $S_1, S_2 : X \to 2^X$, $T : X \times X \to 2^X$, $F : T(X \times X) \times X \times X \to 2^Z$ and $G : T(X \times X) \times X \times X \to 2^Z$ be correspondences with nonempty values. Assume that:*

i) S_2^{-1} *is open-valued and for each* $u \in X$*, the set* $\{x \in X : \exists t \in T(x,u), F(t,u,x) \nsubseteq G(t,x,x)\}$ *is open;*

ii) S_2 *is convex valued and for each* $x \in X$*, the set* $\{u \in X : \exists t \in T(x,u), F(t,u,x) \nsubseteq G(t,x,x)\}$ *is convex;*

iii) if $A = \{x \in X :$ *there exist* $u \in S_2(x)$ *and* $t \in T(x,u)$ *such that* $F(t,u,x) \nsubseteq G(t,x,x)\}$*, then, for each* $N \in \langle A \rangle$*,* $(coN \setminus N) \cap A = \emptyset$*;*

iv) $\{x \in X : x \in S_2(x)\} = \emptyset$*;*

v) there exists $M \in \langle A \rangle$ *such that* $\bigcup_{u \in M}[\{x \in X : \exists t \in T(x,u), F(t,u,x) \nsubseteq G(t,x,x)\} \cap S_2^{-1}(u)] = X$ or $\bigcup_{u \in M}[(X \setminus A) \cap \ S_1^{-1}(u)] = X$.

Then, there exists $x^* \in S_1(x^*)$ *such that* $\forall y \in S_2(x^*)$*,* $\forall t^* \in T(x^*,y)$*,* $F(t^*,y,x^*) \subseteq G(t^*,x^*,x^*)$*.*

Proof. Let $P, G : X \to 2^X$ be defined by

$P(x) = \{u \in X : \exists t \in T(x,u), F(t,u,x) \nsubseteq G(t,x,x)\}$, for each $x \in X$ and

$G(x) = S_2(x) \cap P(x)$, for each $x \in X$.

We are going to show that there exists $x^* \in X$ such that $x^* \in S_1(x^*)$ and $S_2(x^*) \cap P(x^*) = \emptyset$.

We consider two cases.

Case I.

$A = \{x \in X : P(x) \cap K(x) \neq \emptyset\} = \{x \in X : G(x) \neq \emptyset\}$ is nonempty.

The correspondence $G : A \to 2^X$, defined by $G(x) = S_2(x) \cap P(x)$ for each $x \in A$, is nonempty valued on A. We note that for each $u \in X$, $G^{-1}(u) = P^{-1}(u) \cap S_2^{-1}(u)$ is a convex set since it is an intersection of convex sets.

Further, let us define the correspondences $H, L : X \to 2^X$ by

$$H(x) = \begin{cases} G(x), & \text{if } x \in A; \\ \emptyset, & \text{otherwise} \end{cases} = \begin{cases} S_2(x) \cap P(x), & \text{if } x \in A; \\ \emptyset, & \text{otherwise.} \end{cases}$$

According to i) and ii), H is open convex valued.

$$L(x) = \begin{cases} G(x), & \text{if } x \in A; \\ S_1(x), & \text{otherwise.} \end{cases}$$

For each $u \in X$,

$$\begin{aligned} H^{-1}(u) &= \{x \in X : u \in H(x)\} = \\ &= \{x \in A : u \in G(x)\} = \\ &= \{x \in A : u \in S_2(x) \cap P(x)\} \\ &= A \cap P^{-1}(u) \cap S_2^{-1}(u) = \\ &= P^{-1}(u) \cap S_2^{-1}(u). \end{aligned}$$

Since for each $u \in X$, $S_2^{-1}(u)$ and $P^{-1}(u)$ are open, then, $H^{-1}(u)$ is open.

Assumption v) implies that there exists $M \in \langle A \rangle$ such that $\bigcup_{x \in M} H^{-1}(x) = X$.

Let us define $Q : X \to 2^X$ by $Q(x) := X \setminus H^{-1}(x)$ for each $x \in X$.

Then, Q is closed-valued and $\bigcap_{x \in M} Q(x) = X \setminus \bigcup_{x \in M} H^{-1}(x) = \emptyset$.

According to Lemma 4, we can conclude that Q is not a KKM correspondence. Thus, there exists $N \in \langle X \rangle$ such that co$N \subsetneq Q(N) = \bigcup_{x \in N}(X \setminus H^{-1}(x))$.

Hence, there exists $x^* \in$co$N$ with the property that $x^* \in H^{-1}(x)$ for each $x \in N$. Therefore, there exists $x^* \in$co$N$ such that $x^* \in H^{-1}(x)$ for each $x \in N$, which implies $N \subset H(x^*)$. Further, it is true that co$N \subset$co$H(x^*) \subset L(x^*)$. Consequently, $x^* \in$co$N \subset$co$H(x^*) \subset L(x^*)$, which means that $x^* \in L(x^*)$, that is, $x^* \in$co$N$ is a fixed point for L.

We notice that, according to *iv*), $x \notin S_2(x)$ for each $x \in X$, and then, $x^* \notin A$. This fact is possible since $x^* \in$co$N$ and assumption iii) asserts that (co$N \smallsetminus N) \cap A = \emptyset$. Therefore, $x^* \in S_1(x^*)$ and $G(x^*) = S_2(x^*) \cap P(x^*) = \emptyset$.

Consequently, there exists $x^* \in X$ such that $x^* \in S_1(x^*)$ and $\forall y \in S_2(x^*)$, $\forall t^* \in T(x^*, y)$, $F(t^*, y, x^*) \subseteq G(t^*, x^*, x^*)$.

Case II.

$A = \{x \in X : P(x) \cap S_2(x) \neq \emptyset\} = \{x \in X : G(x) \neq \emptyset\}=\emptyset$.

In this case, $G(x) = \emptyset$ for each $x \in X$.

Let us define $Q : X \to 2^X$ by $Q(x) := X \backslash S_1(x)$ for each $x \in X$. The proof follows the same line as above and we obtain that there exists $x^* \in X$ such that $x^* \in S_1(x^*)$.

Obviously, $G(x^*) = \emptyset$. Consequently, the conclusion holds in Case II.

Remark 8 Assumption i) can be replaced with

i') S_2^{-1} is closed-valued and for each $u \in X$, the set $\{x \in X : \exists t \in T(x, u), F(t, u, x) \nsubseteq G(t, x, x)\}$ is closed.

In this case, Q is open-valued.

Theorem 19 can be stated as follows.

Theorem 19 *Let Z be a Hausdorff topological vector space and let X be a nonempty compact convex subset of a topological vector space E. Let $S_1, S_2 : X \to 2^X$, $T : X \times X \to 2^X$, $F : T(X \times X) \times X \times X \to 2^Z$ and $G : T(X \times X) \times X \times X \to 2^Z$ be correspondences with nonempty values. Assume that:*

i) S_2^{-1} is open-valued and for each $u \in X$, the set $\{x \in X : \exists t \in T(x, u), F(t, u, x) \cap G(t, x, x) = \emptyset\}$ is open;

ii) S_2 is convex valued and for each $x \in X$, the set $\{u \in X : \exists t \in T(x, u), F(t, u, x) \cap G(t, x, x) = \emptyset\}$ is convex;

iii) if $A = \{x \in X :$ there exist $u \in S_2(x)$ and $t \in T(x, u)$ such that $F(t, u, x) \cap G(t, x, x) = \emptyset\}$, then, for each $N \in \langle A \rangle$, (co$N \smallsetminus N) \cap A = \emptyset$;

iv) $\{x \in X : x \in S_2(x)\} = \emptyset$;

v) there exists $M \in \langle A \rangle$ such that $\bigcup_{u \in M}[\{x \in X : \exists t \in T(x, u), F(t, u, x) \cap G(t, x, x) = \emptyset\} \cap S_2^{-1}(u)] = X$ or $\bigcup_{u \in M}[(X \backslash A) \cap\ S_1^{-1}(u)] = X$.

Then, there exists $x^* \in S_1(x^*)$ such that $\forall y \in S_2(x^*)$, $\forall t^* \in T(x^*, y)$, $F(t^*, y, x^*) \cap G(t^*, x^*, x^*) = \emptyset$.

Proof. Let $P, G : X \to 2^X$ be defined by

$P(x) = \{u \in X : \exists t \in T(x, u), F(t, u, x) \cap G(t, x, x) = \emptyset\}$, for each $x \in X$ and

$G(x) = S_2(x) \cap P(x)$, for each $x \in X$.

We are going to show that there exists $x^* \in X$ such that $x^* \in S_1(x^*)$ and $S_2(x^*) \cap P(x^*) = \emptyset$.

The rest of the proof follows a similar line as the proof of Theorem 18.

Theorem 20 *Let Z be a Hausdorff topological vector space and let X be a nonempty compact convex subset of a topological vector space E. Let $S_1, S_2 : X \to 2^X$, $T : X \times X \to 2^X$, $F : T(X \times X) \times X \times X \to 2^Z$ and $G : T(X \times X) \times X \times X \to 2^Z$ be correspondences with nonempty values. Assume that:*

i) S_2 is open-valued and for each $x \in X$, the set $\{u \in X : \exists t \in T(x,u), F(t,u,x) \nsubseteq G(t,x,x)\}$ is open;

ii) if $A = \{x \in X :$ there exist $u \in S_2(x)$ and $t \in T(x,u)$ such that $F(t,u,x) \nsubseteq G(t,x,x)\}$, then, for each $N \in \langle A \rangle$, $(coN \setminus N) \cap A = \emptyset$;

iii) $\{x \in X : x \in S_2(x)\} = \emptyset$;

iv) there exists $M \in \langle A \rangle$ such that $\bigcup_{x \in M}[\{u \in X : \exists t \in T(x,u), F(t,u,x) \nsubseteq G(t,x,x)\} \cap S_2(x)] = X$;

v) $S_2^{-1} : X \to 2^X$ is convex valued and for each $u \in X$, the set $\{x \in X : \exists t \in T(x,u), F(t,u,x) \nsubseteq G(t,x,x)\}$ is convex;

Then, there exists $x^* \in S_1(x^*)$ such that $\forall y \in S_2(x^*)$, $\forall t^* \in T(x^*,y)$, $F(t^*,y,x^*) \subseteq G(t^*,x^*,x^*)$.

Proof. Let $P, G : X \to 2^X$ be defined by

$P(x) = \{u \in X : \exists t \in T(x,u), F(t,u,x) \nsubseteq G(t,x,x)\}$, for each $x \in X$ and

$G(x) = S_2(x) \cap P(x)$, for each $x \in X$.

We are going to show that there exists $x^* \in X$ such that $x^* \in S_1(x^*)$ and $S_2(x^*) \cap P(x^*) = \emptyset$.

We consider two cases.

Case I.

$A = \{x \in X : S_2(x) \cap P(x) \neq \emptyset\} = \{x \in X : G(x) \neq \emptyset\}$ is nonempty.

The correspondence $G : A \to 2^X$, defined by $G(x) = S_2(x) \cap P(x)$ for each $x \in A$, is nonempty valued on A. We note that for each $u \in X$, $G^{-1}(u) = P^{-1}(u) \cap S_2^{-1}(u)$ is a convex set as intersection of convex sets.

Further, let us define the correspondences $H, L : X \to 2^X$ by

$$H(x) = \begin{cases} G(x), & \text{if } x \in A; \\ \emptyset, & \text{otherwise} \end{cases}$$

$$L(x) = \begin{cases} G(x), & \text{if } x \in A; \\ S_1(x), & \text{otherwise} \end{cases} = \begin{cases} S_2(x) \cap P(x), & \text{if } x \in A; \\ S_1(x), & \text{otherwise.} \end{cases}$$

According to i), H is open-valued and according to v),for each $u \in X$, $P^{-1}(u)$ is convex.

For each $u \in X$,

$$\begin{aligned} H^{-1}(u) &= \{x \in X : u \in H(x)\} = \\ &= \{x \in A : u \in G(x)\} = \\ &= \{x \in A : u \in S_2(x) \cap P(x)\} \\ &= A \cap P^{-1}(u) \cap S_2^{-1}(u) = \\ &= P^{-1}(u) \cap S_2^{-1}(u). \end{aligned}$$

$L^{-1}(u) = \{x \in X : u \in L(x)\} =$
$= \{x \in A : u \in G(x)\} \cup \{x \in X \setminus A : u \in S_1(x)\} =$
$= \{x \in A : u \in S_2(x) \cap P(x)\} \cup \{x \in X \setminus A : u \in S_1(x)\}$
$= (A \cap P^{-1}(u) \cap S_2^{-1}(u) \cup [(X \backslash A) \cap S_1^{-1}(u)] =$
$= [P^{-1}(u) \cap S_2^{-1}(u)] \cup [(X \backslash A) \cap S_1^{-1}(u)]$

Since for each $u \in X$, $S_2^{-1}(u)$ and $P^{-1}(u)$ are convex, then, $H^{-1}(u)$ is convex. Therefore, co$H^{-1}(u) \subset L^{-1}(u)$ for each $u \in X$.

Assumption iv) implies that there exists $M \in \langle A \rangle$ such that $\bigcup_{x \in M} H(x) = X$.

Let us define $Q : X \to 2^X$ by $Q(x) := X \backslash H(x)$ for each $x \in X$.

Then, Q is closed-valued and $\bigcap_{x \in M} Q(x) = X \backslash \bigcup_{x \in M} H(x) = \emptyset$.

According to Lemma 4, we can conclude that Q is not a KKM correspondence. Thus, there exists $N \in \langle X \rangle$ such that co$N \subsetneq Q(N) = \bigcup_{x \in N} (X \backslash H(x))$.

Hence, there exists $x^* \in$co$N$ with the property that $x^* \in H(x)$ for each $x \in N$. Therefore, there exists $x^* \in$co$N$ such that $x^* \in H(x)$ for each $x \in N$, which implies $N \subset H^{-1}(x^*)$. Further, it is true that co$N \subset$co$H^{-1}(x^*) \subset L^{-1}(x^*)$. Consequently, $x^* \in$co$N \subset$co$H^{-1}(x^*) \subset L^{-1}(x^*)$, which means that $x^* \in L(x^*)$, that is, $x^* \in$co$N$ is a fixed point for L.

We notice that, according to iii), $x \notin S_2(x)$ for each $x \in X$, and then, $x^* \notin A$. This fact is possible since $x^* \in$co$N$ and assumption iii) asserts that (co$N \setminus N) \cap A = \emptyset$. Therefore, $x^* \in S_1(x^*)$ and $G(x^*) = S_2(x^*) \cap P(x^*) = \emptyset$.

Consequently, there exists $x^* \in S_1(x^*)$ such that $\forall y \in S_2(x^*)$, $\forall t^* \in T(x^*, y)$, $F(t^*, y, x^*) \subseteq G(t^*, x^*, x^*)$.

Case II.

$A = \{x \in X : P(x) \cap S_2(x) \neq \emptyset\} = \{x \in X : G(x) \neq \emptyset\} = \emptyset$.

In this case, $G(x) = \emptyset$ for each $x \in X$.

Let us define $Q : X \to 2^X$ by $Q(x) := X \backslash S_1(x)$ for each $x \in X$. The proof follows the same line as above and we obtain that there exists $x^* \in X$ such that $x^* \in S_1(x^*)$.

Obviously, $G(x^*) = \emptyset$. Consequently, the conclusion holds in Case II.

Remark 9 Assumption i) can be replaced with

i') S_2 is closed-valued and for each $x \in X$, the set $\{u \in X : \exists t \in T(x, u), F(t, u, x) \nsubseteq G(t, x, x)\}$ is closed.

In this case, Q is open-valued.

Now, we are proving the existence of solutions for a general vector variational inclusion problem concerning correspondences under new assumptions.

Theorem 21 *Let Z be a Hausdorff topological vector space and let X be a nonempty compact convex subset of a topological vector space E. Let $S_1, S_2 : X \to 2^X$, $T : X \times X \to 2^X$ and $F : T(X \times X) \times X \times X \to 2^Z$ and $G : T(X \times X) \times X \times X \to 2^Z$ be correspondences with nonempty values. Assume that:*

i) S_1, S_2 are open-valued and for each $(x, y) \in X \times X$ with the property that $\exists t \in T(x, y)$ such that $F(t, y, x) \nsubseteq G(t, x, x)\}$, there exists $z = z_{x,y} \in X$ such that $y \in int_X\{u \in X : \exists t \in T(z_{x,y}, u), F(t, u, z_{x,y}) \nsubseteq G(t, z_{x,y}, z_{x,y})\}$;

ii) there exists $M \in \langle X\rangle$ such that
$\bigcup_{x\in M\cap A}[\bigcup_{\{y\in S_2(x),\ t\in T(x,y):\ F(t,y,x)\nsubseteq G(t,x,x)\}}(int_X\{u \in X : \exists t \in T(z_{x,y}, u),$ $F(t, u, z_{x,y}) \nsubseteq G(t, z_{x,y}, z_{x,y})\} \cap S_2(x)] \bigcup \bigcup_{x\in M\setminus A} S_1(x) = X$, *where*
$A = \{x \in X$: *there exist* $u \in S_2(x)$, $t \in T(x, u)$ *such that* $F(t, u, x) \nsubseteq G(t, x, x)\}$;

iii) $\{x \in X : x \in S_2(x)\} = \emptyset$;

iv) for each $u \in X$, the set M_u is convex, where
$M_u = \{x \in X : \exists t \in T(x, u), F(t, u, x) \nsubseteq G(t, x, x)\} \cup [(X\setminus A) \cap (S_1^{-1}(u)]$;

v) for each $x \in A$,
$\bigcup_{\{y\in S_2(x),\ t\in T(x,y):\ F(t,y,x)\nsubseteq G(t,x,x)\}}(int_X\{u \in X : \exists t \in T(z_{x,y}, u), F(t, u, z_{x,y}) \nsubseteq G(t, z_{x,y}, z_{x,y})\} \subset \{u \in X : \exists t \in T(x, u), F(t, u, x) \nsubseteq G(t, x, x)\}$;

Then, there exists $x^* \in S_1(x^*)$ such that $\forall y \in S_2(x^*)$, $\forall t^* \in T(x^*, y)$, $F(t^*, y, x^*) \subseteq G(t^*, x^*, x^*)$.

Proof. Let $P, G : X \to 2^X$ be defined by

$P(x) = \{u \in X : \exists t \in T(x, u), F(t, u, x) \nsubseteq G(t, x, x)\}$, for each $x \in X$ and

$G(x) = S_2(x) \cap P(x)$, for each $x \in X$.

We are going to show that there exists $x^* \in X$ such that $x^* \in S_1(x^*)$ and $S_2(x^*) \cap P(x^*) = \emptyset$.

We consider two cases.

Case I.

$A = \{x \in X : P(x) \cap S_2(x) \neq \emptyset\} = \{x \in X : G(x) \neq \emptyset\}$ is nonempty.

The correspondence $G : A \to 2^X$, defined by $G(x) = S_2(x) \cap P(x)$ for each $x \in A$, is nonempty on A.

Further, let us define the correspondences $H, L : X \to 2^X$ by

$H(x) = \begin{cases} G(x), & \text{if } x \in A; \\ S_1(x), & \text{otherwise} \end{cases}$ and

$L(x) = \begin{cases} P(x), & \text{if } x \in A; \\ S_1(x), & \text{otherwise} \end{cases}$

For each $u \in X$,

$$\begin{aligned} L^{-1}(u) &= \{x \in X : u \in L(x)\} = \\ &= \{x \in A : u \in P(x)\} \cup \{x \in X \setminus A : u \in S_1(x)\} = \\ &= (A \cap P^{-1}(u)) \cup [(X\setminus A)\cap S_1^{-1}(u)] = \\ &= P^{-1}(u) \cup [(X\setminus A) \cap S_1^{-1}(u)]. \end{aligned}$$

According to i), G is transfer open-valued. Assumptions i) and ii) imply that there exists $M \in \langle X\rangle$ and for each $x \in M$ and $y \in H(x)$, there exists $z_{x,y} \in X$ such that $y \in \mathrm{int}_X H(z_{x,y}) \cap H(x)$ and $\bigcup_{x\in M}(\bigcup_{y\in H(x)} \mathrm{int}_X H(z_{x,y})) = X$. In addition, assumption v) implies $\bigcup_{y\in H(x)} \mathrm{int}_X H(z_{x,y}) \subseteq P(x)$ for each $x \in A$. We note that if $x \in X\setminus A$, then $H(x) = S_1(x)$ is open and $y \in H(x)$ implies $z_{x,y} = x$ and $y \in H(z_{x.y}) = \mathrm{int} H(x)$. In this last case it is obvious that $\bigcup_{y\in H(x)} \mathrm{int}_X H(z_{x,y}) = \bigcup_{y\in H(x)} \mathrm{int}_X H(x) = \bigcup_{y\in H(x)} H(x) = H(x)$.

Let us define $Q : X \to 2^X$ by $Q(x) := X \setminus \bigcup_{y \in H(x)} \text{int}_X H(z_{x,y})$ for each $x \in H$.

Then, Q is closed-valued and $\bigcap_{x \in M} Q(x) = X \setminus \bigcup_{x \in M} (\bigcup_{y \in H(x)} \text{int}_X H(z_{x,y})) = \emptyset$.

According to Lemma 4, we can conclude that Q is not a KKM correspondence. Thus, there exists $N \in \langle X \rangle$ such that co$N \subsetneq Q(N) = \bigcup_{x \in N} (X \setminus \bigcup_{y \in H(x)} \text{int}_X H(z_{x,y}))$.

Hence, there exists $x^* \in$co$N$ with the property that $x^* \in \bigcup_{y \in H(x)} \text{int}_X H(z_{x,y})$ for each $x \in N$. If $x \in A$, $\bigcup_{y \in H(x)} \text{int}_X H(z_{x,y}) \subset P(x)$ and if $x \in X \setminus A$, $\bigcup_{y \in H(x)} \text{int}_X H(z_{x,y}) = H(x)$. Therefore, there exists $x^* \in$co$N$ such that $x^* \in L(x)$ for each $x \in N$, which implies $N \subset L^{-1}(x^*)$. Further, it is true that co$N \subset$co$L^{-1}(x^*) = L^{-1}(x^*)$. Consequently, $x^* \in$co$N \subset$co$L^{-1}(x^*) = L^{-1}(x^*)$, which means that $x^* \in L(x^*)$. We notice that, according to ii), $x \notin S_2(x)$ for each $x \in X$, and then, $x^* \notin A$. Therefore, $x^* \in S_1(x^*)$ and $G(x^*) = S_2(x^*) \cap P(x^*) = \emptyset$.

Consequently, there exists $x^* \in S_1(x^*)$ such that $\forall y \in S_2(x^*)$, $\forall t^* \in T(x^*, y)$, $F(t^*, y, x^*) \subseteq G(t^*, x^*, x^*)$.

Case II.

$A = \{x \in X : P(x) \cap S_2(x) \neq \emptyset\} = \{x \in X : G(x) \neq \emptyset\}$=$\emptyset$.

In this case, $G(x) = \emptyset$ for each $x \in X$. Let us define $Q : X \to 2^X$ by $Q(x) := X \setminus S_1(x)$ for each $x \in X$. The proof follows the same line as above and we obtain that there exists $x^* \in X$ such that $x^* \in S_1(x^*)$.

Obviously, $G(x^*) = \emptyset$. Consequently, the conclusion holds in Case II.

Now, we are establishing Theorem 22.

Theorem 22 *Let Z be a Hausdorff topological vector space and let X be a nonempty compact convex subset of a topological vector space E. Let $S_1, S_2 : X \to 2^X$, $T : X \times X \to 2^X$, $F : T(X \times X) \times X \times X \to 2^Z$ and $G : T(X \times X) \times X \times X \to 2^Z$ be correspondences with nonempty values. Assume that:*

i) S_1, S_2 are open-valued and for each $(x, y) \in X \times X$ with the property that $\exists t \in T(x, y)$ such that $F(t, y, x) \cap G(t, x, x) = \emptyset\}$, there exists $z = z_{x,y} \in X$ such that $y \in int_X \{u \in X : \exists t \in T(z_{x,y}, u), F(t, u, z_{x,y}) \cap G(t, z_{x,y}, z_{x,y}) = \emptyset\}$;

ii) there exists $M \in \langle X \rangle$ such that
$\bigcup_{x \in M \cap A} [\bigcup_{\{y \in S_2(x),\ t \in T(x,y):\ F(t,y,x) \nsubseteq G(t,x,x)\}} (int_X \{u \in X : \exists t \in T(z_{x,y}, u), F(t, u, z_{x,y}) \cap G(t, z_{x,y}, z_{x,y}) = \emptyset\} \cap S_2(x)] \bigcup \bigcup_{x \in M \setminus A} S_1(x) = X$, where $A = \{x \in X :$ there exist $u \in S_2(x)$, $t \in T(x, u)$ such that $F(t, u, x) \cap G(t, x, x) = \emptyset\}$;

iii) $\{x \in X : x \in S_2(x)\} = \emptyset$;

iv) for each $u \in X$, the set M_u is convex, where

$M_u = \{x \in X : \exists t \in T(x, u), F(t, u, x) \nsubseteq G(t, x, x)\} \cup [(X \setminus A) \cap (S_1^{-1}(u)]$;

v) for each $x \in A$, $\bigcup_{y \in S_2(x),\ \exists t \in T(x,y) \text{ such that } F(t,y,x) \nsubseteq G(t,x,x)} (int_X \{u \in X : \exists t \in T(z_{x,y}, u), F(t, u, z_{x,y}) \cap G(t, z_{x,y}, z_{x,y}) = \emptyset\} \subset$

$\{u \in X : \exists t \in T(x,u), F(t,u,x) \cap G(t,x,x) = \emptyset\}$;

Then, there exists $x^* \in S_1(x^*)$ *such that* $\forall y \in S_2(x^*)$, $\forall t^* \in T(x^*,y)$, $F(t^*,y,x^*) \cap G(t^*,x^*,x^*) \neq \emptyset$.

We establish sufficient conditions which assure the existence of solutions for a general vector variational inclusion problem.

Theorem 23 *Let* Z *be a Hausdorff topological vector space and let* X *be a nonempty compact convex subset of a topological vector space* E. *Let* $S_1, S_2 : X \to 2^X$, $T : X \times X \to 2^X$, $F : T(X \times X) \times X \times X \to 2^Z$ *and* $G : T(X \times X) \times X \times X \to 2^Z$ *be correspondences with nonempty values. Assume that:*

i) S_1, S_2 *are open-valued and for each* $(x,y) \in X \times X$ *with the property that* $\exists t \in T(x,y)$ such that $F(t,y,x) \nsubseteq G(t,x,x)\}$, *there exists* $z = z_{x,y} \in X$ *such that* $y \in int_X\{u \in X : \exists t \in T(z_{x,y},u), F(t,u,z_{x,y}) \nsubseteq G(t,z_{x,y},z_{x,y})\}$;

ii) for each $x \in X$ *and* $t \in T(x,x), F(t,x,x) \subseteq G(t,x,x)$;

iii) if $A = \{x \in X :$ *there exist* $u \in S_2(x)$ *and* $t \in T(x,u)$ *such that* $F(t,u,x) \nsubseteq G(t,x,x)\}$, *then for each* $N \in \langle A \rangle$, $(coN \setminus N) \cap A = \emptyset$;

iv) there exists $M \in \langle A \rangle$ *such that*
$\bigcup_{x\in M}[\bigcup_{y\in S_2(x),\ \exists t\in T(x,y) \text{ such that } F(t,y,x)\nsubseteq G(t,x,x)}(int_X\{u \in X : \exists t \in T(z_{x,y},u),$ $F(t,u,z_{x,y}) \nsubseteq G(t,z_{x,y},z_{x,y})\} \cap S_2(x)] = X$;

iv) for each $u \in X$, *the set* M_u *is convex, where*
$M_u = \{x \in X : \exists t \in T(x,u), F(t,u,x) \nsubseteq G(t,x,x)\}$;

v) for each $x \in A$,
$\bigcup_{\{y\in S_2(x),\ t\in T(x,y):\ F(t,y,x)\nsubseteq G(t,x,x)\}}(int_X\{u \in X : \exists t \in T(z_{x,y},u), F(t,u,z_{x,y}) \nsubseteq$ $G(t,z_{x,y},z_{x,y})\} \cap S_2(x) \subset \{u \in X : \exists t \in T(x,u), F(t,u,x) \nsubseteq G(t,x,x)\}$;

Then, there exists $x^* \in S_1(x^*)$ *such that* $\forall y \in S_2(x^*)$, $\forall t^* \in T(x^*,y)$, $F(t^*,y,x^*) \subseteq G(t^*,x^*,x^*)$.

Proof. Let $P, G : X \to 2^X$ be defined by

$P(x) = \{u \in X : \exists t \in T(x,u), F(t,u,x) \nsubseteq G(t,x,x)\}$, for each $x \in X$ and

$G(x) = S_2(x) \cap P(x)$, for each $x \in X$.

We are going to show that there exists $x^* \in X$ such that $x^* \in S_1(x^*)$ and $S_2(x^*) \cap P(x^*) = \emptyset$.

We consider two cases.

Case I.

$A = \{x \in X : P(x) \cap S_2(x) \neq \emptyset\} = \{x \in X : G(x) \neq \emptyset\}$ is nonempty.

The correspondence $G : A \to 2^X$, defined by $G(x) = S_2(x) \cap P(x)$ for each $x \in A$, is nonempty valued on A.

Further, let us define the correspondences $H, L, M : X \to 2^X$ by

$H(x) = \begin{cases} G(x), & \text{if } x \in A; \\ \emptyset, & \text{otherwise} \end{cases}$,

$M(x) = \begin{cases} P(x), & \text{if } x \in A; \\ \emptyset, & \text{otherwise} \end{cases}$ and

$L(x) = \begin{cases} P(x), & \text{if } x \in A; \\ S_1(x), & \text{otherwise.} \end{cases}$

For each $u \in X$,
$M^{-1}(u) = \{x \in X : u \in M(x)\} =$
$= \{x \in A : u \in P(x)\} =$
$= A \cap P^{-1}(u) =$
$= P^{-1}(u) = M_u$.
$L^{-1}(u) = \{x \in X : u \in L(x)\} =$
$= \{x \in A : u \in P(x)\} \cup \{x \in X \setminus A : u \in S_1(x)\} =$
$= (A \cap P^{-1}(u)) \cup [(X \backslash A) \cap (S_1^{-1}(u)] =$
$= P^{-1}(u) \cup [(X \backslash A) \cap (S_1^{-1}(u)]$.

According to assumption iv), $P^{-1}(u)$ is convex for each $u \in X$. Then, $\mathrm{co}M^{-1}(u) =$
$=\mathrm{co}P^{-1}(u) = P^{-1}(u) \subset L^{-1}(u)$ for each $u \in X$.

According to i), G is transfer open-valued. Assumptions i) and iii) imply that there exists $M \in \langle A \rangle$ and for each $x \in M$ and $y \in H(x)$, there exists $z_{x,y} \in X$ such that $y \in \mathrm{int}_X H(z_{x,y}) \cap H(x)$ and $\bigcup_{x \in M}(\bigcup_{y \in H(x)} \mathrm{int}_X H(z_{x,y})) = X$. In addition, assumption vi) implies $\bigcup_{y \in H(x)} \mathrm{int}_X H(z_{x,y}) \subseteq P(x)$ for each $x \in A$.

Let us define $Q : X \to 2^X$ by $Q(x) := X \backslash \bigcup_{y \in H(x)} \mathrm{int}_X H(z_{x,y})$ for each $x \in X$.

Then, Q is closed-valued and $\bigcap_{x \in M} Q(x) = X \backslash \bigcup_{x \in M}(\bigcup_{y \in H(x)} \mathrm{int}_X H(z_{x,y})) = \emptyset$, where $M \subset A$.

According to Lemma 4, we conclude that Q is not a KKM correspondence. Thus, there exists $N \in \langle A \rangle$ such that $\mathrm{co}N \subsetneq Q(N) = \bigcup_{x \in N}(X \backslash \bigcup_{y \in H(x)} \mathrm{int}_X H(z_{x,y}))$.

Hence, there exists $x^* \in \mathrm{co}N$ with the property that $x^* \in \bigcup_{y \in H(x)} \mathrm{int}_X H(z_{x,y}) \subset M(x)$ for each $x \in N$. Therefore, there exists $x^* \in \mathrm{co}N$ such that $x^* \in M(x)$ for each $x \in N$, which implies $N \subset M^{-1}(x^*)$. Further, it is true that $\mathrm{co}N \subset \mathrm{co}M^{-1}(x^*) \subset L^{-1}(x^*)$. Consequently, $x^* \in \mathrm{co}N \subset \mathrm{co}M^{-1}(x^*) \subset L^{-1}(x^*)$, which means that $x^* \in L(x^*)$. We notice that, according to ii), $x \notin P(x)$ for each $x \in X$, and then, $x^* \notin A$. This fact is possible since $x^* \in \mathrm{co}N$ and assumption iii) asserts that $(\mathrm{co}N \setminus N) \cap A = \emptyset$. Therefore, $x^* \in S_1(x^*)$ and $G(x^*) = S_2(x^*) \cap P(x^*) = \emptyset$.

Consequently, there exists $x^* \in S_1(x^*)$ such that $\forall y \in S_2(x^*)$, $\forall t^* \in T(x^*, y)$, $F(t^*, y, x^*) \subseteq G(t^*, x^*, x^*)$.

Case II.

$A = \{x \in X : P(x) \cap S_2(x) \neq \emptyset\} = \{x \in X : G(x) \neq \emptyset\} = \emptyset$.

In this case, $G(x) = \emptyset$ for each $x \in X$. Let us define $Q : X \to 2^X$ by $Q(x) := X \backslash S_1(x)$ for each $x \in X$. The proof follows the same line as above and we obtain that there exists $x^* \in X$ such that $x^* \in S_1(x^*)$.

Obviously, $G(x^*) = \emptyset$. Consequently, the conclusion holds in Case II.

The next result can be obtained similarly as Theorem 23.

Theorem 24 *Let Z be a Hausdorff topological vector space and let X be a nonempty compact convex subset of a topological vector space E. Let $S_1, S_2 : X \to 2^X$, $T : X \times X \to 2^X$, $F : T(X \times X) \times X \times X \to 2^Z$ and $G : T(X \times X) \times X \times X \to 2^Z$ be correspondences with nonempty values. Assume that:*

i) S_1, S_2 are open-valued and for each $(x,y) \in X \times X$ with the property that $\exists t \in T(x,y)$ such that *$F(t,y,x) \cap G(t,x,x)\} = \emptyset$, there exists $z = z_{x,y} \in X$ such that $y \in int_X\{u \in X : \exists t \in T(z_{x,y},u), F(t,u,z_{x,y}) \cap G(t,z_{x,y},z_{x,y})\} = \emptyset$;*

ii) for each $x \in X$ and $t \in T(x,x), F(t,x,x) \subseteq G(t,x,x)$;

iii) if $A = \{x \in X :$ *there exist $u \in S_2(x)$ and $t \in T(x,u)$ such that $F(t,u,x) \cap G(t,x,x) = \emptyset\}$, then for each $N \in \langle A \rangle$, $(coN \setminus N) \cap A = \emptyset$;*

iv) there exists $M \in \langle A \rangle$ such that
$\bigcup_{x\in M}[\bigcup_{\{y\in S_2(x),\ t\in T(x,y):\ F(t,y,x)\nsubseteq G(t,x,x)\}}(int_X\{u \in X : \exists t \in T(z_{x,y},u), F(t,u,z_{x,y})\cap G(t,z_{x,y},z_{x,y}) = \emptyset\} \cap S_2(x)] = X$;

iv) for each $u \in X$, the set $\{x \in X : \exists t \in T(x,u), F(t,u,x) \cap G(t,x,x) = \emptyset\}$ is convex;

v) for each $x \in A$,
$\bigcup_{\{y\in S_2(x),\ t\in T(x,y):\ F(t,y,x)\nsubseteq G(t,x,x)\}}(int_X\{u \in X : \exists t \in T(z_{x,y},u), F(t,u,z_{x,y})\cap G(t,z_{x,y},z_{x,y}) = \emptyset\} \cap S_2(x) \subset \{u \in X : \exists t \in T(x,u), F(t,u,x) \cap G(t,x,x) = \emptyset\}$;

Then, there exists $x^* \in S_1(x^*)$ such that $\forall y \in S_2(x^*)$, $\forall t^* \in T(x^*,y)$, $F(t^*,y,x^*) \cap G(t^*,x^*,x^*) \neq \emptyset$.:

5 Concluding remarks

In this paper, we have introduced T-properly quasi-convex correspondences and T-properly quasi-convex sets. We have given some examples, as well. We have used these notions to obtain coincidence-like theorems, to solve vector equilibrium problems and to establish a generalized minimax inequality, which is new in literature. We have also proved the existence of solutions for vector variational inclusion problems, by applying the KKM principle. Our research extends on some results which exist in literature. This study can be continued by considering abstract convex spaces and generalized KKM theorems.

References

1. Q. H. Ansari, A. P. Farajzadeh, S. Schaible, Existence of solutions of strong vector equilibrium problems, *Taiwanese Journal of Mathematics* 16, *1*, (2012) 165-178
2. M. Balaj, Coincidence and maximal element theorems and their applications to generalized equilibrium problems and minimax inequalities, Nonlinear Anal. 68 (2008), 3962-3971.
3. M. Balaj, Solution existence for two types of mixed variational relation problems, Applicable Analysis, DOI: 10.1080/00036811.2015.1006204
4. G. Bigi, A. Capătă and G. Kassay, Existence results for strong vector equilibrium problems and their applications, *Optimization,* 61, *5* (2012), 567-583.
5. M. Castellani, M. Giuli, Refinements of existence results for relaxed quasimonotone equilibrium problems, Journal of Global Optimization , 57 (2013), 1213-1227

6. A. Dhara and D.T. Luc, A solution method for linear variational relation problems, J. Global Optimization 59 (2014) 729-756
7. X. P. Ding, Collective fixed points, generalized games and systems of generalized quasi-variational inclusion problems in topological spaces, Nonlinear Analysis, 73 (2010), 1834-1841.
8. B. V. Dinh, P. G. Hung and L. D. Muu, Bilevel Optimization as a Regularization Approach to Pseudomonotone Equilibrium Problems, Numerical Functional Analysis and Optimization 35 (2014) 539-563
9. K. Fan, Minimax theorems. Proc. Natl. Acad. Sci. U.S.A. 39 (1953) 42-47.
10. K. Fan, A Minimax Inequality and Applications. Inequalities III, Academic Press, New York, NY;1972.
11. K. Fan, Sur un théorème minimax. C. R. Acad. Sci. Paris. 259 (1964), 3925-3928.
12. S.-M. Guu and J. Li, Vector quasi-equilibrium problems: separation, saddle points and error bounds for the solution set, Journal of Global Optimization, 58, (2014), 751-767
13. P. G. Georgiev, T. Tanaka, Ky Fan's Inequality for Set-Valued Maps with Vector-Valued Images. Nonlinear analysis and convex analysis (Kyoto, 2000) Sūrikaisekikenkyūsho Kōkyūroku 1187 (2001), 143-154.
14. N.X. Hai and P.Q. Khanh, The solution existence of general variational inclusion problems, J. Math. Anal. Appl. 328 (2007) 1268-1277.
15. Kim W. K., *Some applications of the Kakutani fixed point theorem,* J. Math. Anal. Appl. 121 (1987), 119-122
16. D. Kuroiowa, T. Tanaka, T. X. D. Ha, On cone convexity of set-valued maps. Non-linear Anal.: Theory, Methods Appl. (30) 1997, 1487-1496.
17. L. J. Lin, Applications of a fixed point theorem in G-convex spaces, Nonlinear Anal. 46 (2001) 601-608.
18. L.J. Lin, Q. H. Ansari and J. Y. Wu, Geometric properties and coincidence theorems with applications to generalized vector equilibrium problems, J. Optim. Theory Appl. 117(2003) 121-137
19. L. J. Lin, W.P. Wan, KKM type theorems and coincidence theorems with applications to the existence of equilibrium, J. Optim. Theory Appl. 123 (2004) 105-122.
20. Y. C. Lin, On generalized vector equilibrium problems, *Nonlinear Analysis* 70 (2009) 1040–1048.
21. L.-J. Lin, Some results on systems of quasi-variational inclusion problems and systems of generalized quasi-variational inclusion problems, Nonlinear Analysis, 72 (2010), 37-49.
22. L.-J. Lin, C.-S. Chuang and S. Al-Homidan, Ekeland type variational principle with applications to quasi-variational inclusion problems, Nonlinear Analysis, 72 (2010), 651-661.
23. S. Park, Generalizations of Ky Fan matching theorems and their applications, J. Math. Anal. Appl. 141 (1989), 164-176.
24. S. Park, Coincidence points and maximal elements of multifunctions on convex spaces, Comment. Math. Univ. Carolina 36 (1995) 57-67.
25. Park S., *The KKM principle implies many fixed point theorems,* Topology and its applications **135 (2004),** 197-206.
26. M. Patriche, Fixed point theorems and applications in theory of games. Fixed Point Theory 2014;15:199-212.
27. M. Patriche, Minimax theorems for set-valued maps without continuity assumptions, Optimization, DOI: 10.1080/02331934.2015.1091822

28. Z. Y. Peng, Y. Zhao and X. M. Yang, Semicontinuity of Approximate Solution Mappings to Parametric Set-Valued Weak Vector Equilibrium Problems, Numerical Functional Analysis and Optimization 36 (2015), 481-500
29. P. H. Sach and N. B. Minh, New Results on Henig Proper Generalized Vector Quasiequilibrium Problems, Numerical Functional Analysis and Optimization, 36 (2015), 387-403
30. Shih M.-H. and Tan K.-K., *Covering theorems of convex sets related to fixed-point theorems*, in "Nonlinear and Convex Analysis-Proc. in Honor of Ky Fan" (B-L. Lin and S. Simons, eds.), Marcel Dekker, New York, 1987, pp. 235-244.
31. C. G. Tian, Generalizations of the KKM theorem and the Ky Fan minimax inequality, with applications to maximal elements, price equilibrium, and complementarity, J. Math. Anal. Appl. 170 (1992) 457-471.
32. G. Tian and J. Zhou. Transfer continuities, generalizations of the Weierstrass and maximum theorems: a full characterization, J. Math. Econ. 24 (1995), 281-303.
33. Y. Zhang, S. J. Li and S. K. Zhu, Minimax Problems for Set-Valued Mappings, Numerical Functional Analysis and Optimization 33 (2012), 239-253
34. H. Zegeye and N. Shahzad, Algorithms for Solutions of Variational Inequalities in the Set of Common Fixed Points of Finite Family of [redacted]-Strictly Pseudo-contractive Mappings, Numerical Functional Analysis and Optimization 36 (2015) 799-816